Domain Decomposition Capabilities for the Mortar Finite Volume Element Methods *

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1 Introduction

Since the introduction of the *mortar method* as a coupling technique between the spectral and finite element methods (see, e.g. [5, 7, 8]), it has become the most important technique in domain decomposition methods for non-matching grids. The active research by the scientific computation community in this field is motivated by its flexibility and great potential for large scale parallel computation (see, e.g. [3]). A good description of the mortar element method can be found in [2, 6, 7, 12]. The nonconforming finite element mortar method has been studied in [7], where optimal order convergence in H^1-norm was demonstrated. Three-dimensional mortar finite element analysis has been given in [5]. Non-mortar mixed finite element approximations for second order elliptic problems have been discussed in [1].

The above mentioned mortar elements are defined on non-matching grids with non-overlapping subdomains. Recently, the overlapping mortar linear finite element method was studied in [9], where several additive Schwarz preconditioners have been proposed and analyzed and extensive numerical examples to support the theoretical results have been reported.

To the authors' best knowledge, there has not been a study for the mortar finite volume element method. In the past 10 years the finite (control) volume method has drawn serious attension both form mathematicians, engineers, and physicists as an attractive solution technique for various applied problems (see, e.g. [4]). Following the notations and the approach of Ben Belgacem [2], we

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extend the mortar technique to the finite volume element methods in two ways: (1) following the traditional mortar approach we propose mortar finite volume elements only on the subdomains with finite elements on the interfaces for Lagrange multipliers; (2) we propose numerical schemes by using finite volume elements on both the subdomains and the interfaces. It has been shown by numerical examples that the latter schemes converge much faster (5-6 times) than the former schemes. In this respect we have found evidence from our numerical experiments and we believe that in the finite volume element methods if r-th order piecewise polynomials are used on the subdomains, then $(r-1)-$th order polynomials should be used on the interface for the Lagrange multipliers. For both types of schemes, we have obtained in [13] optimal order H^1-norm error estimates under the regularity assumption that $u \in H^{1+\tau_k}(\Omega_k)$ for $0 < \tau_k \le 1$ where $\overline{\Omega} = \cup \overline{\Omega}_k$.

2 Notations and Mortar Finite Element Approximation

We shall use the notations from [2]. We break up the initial domain Ω into K non-overlapping subdomains $\{\Omega_k\}_{1\le k \le K}$, which are assumed to be polygonally shaped and arranged in such a way that the intersection of two subdomains $\overline{\Omega}_l \cap \overline{\Omega}_k$ as well as the intersection $\partial\Omega \cap \partial\Omega_k$ is either empty or reduced to a vertex or to a common edge. If two subdomains Ω_k and Ω_l are adjacent, Γ_{kl} is the common interface, and $\mathbf{n}_{kl}$ is the unit normal from Ω_k to Ω_l. Let $\overline{k}$ denote the set of all indices so that kl is meaningful. For any k, let $H^1_*(\Omega_k)$ denote the space $H^1(\Omega_k)$ if the measure of $\partial\Omega_k \cap \partial\Omega$ is zero; otherwise it coincides with the subspace of $H^1(\Omega_k)$ involving all functions whose trace is zero over the set $\partial\Omega_k \cap \partial\Omega$:

$$H^1_*(\Omega_k) = \left\{ v_k \in H^1(\Omega_k) : \quad v_k|_{\partial\Omega_k \cap \partial\Omega} = 0, \text{ if } \operatorname{meas}(\partial\Omega_k \cap \partial\Omega) \neq 0 \right\}.$$

Set the space

$$X = \left\{ v \in L^2(\Omega) : \quad v_k = v|_{\Omega_k} \in H^1_*(\Omega_k) \right\} = \prod_{k=1}^{K} H^1_*(\Omega_k)$$

equipped with the norm: $||u||_X = \left(\sum_{k=1}^K ||v_k||^2_{H^1(\Omega_k)} \right)^{1/2}$. Let

$$H_0(\mathrm{div}, \Omega) = \{ q \in H(\mathrm{div}, \Omega) : \quad \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0 \},$$

where $H(\mathrm{div}, \Omega)$ is the space of all vector-functions in $(L^2(\Omega))^2$ whose weak divergence is in $L^2(\Omega)$. The trace of these function on the boundary $\partial\Omega$ is understood in the appropriate weak sense. The characterization of $H^1_0(\Omega)$ can be made:

$$H^1_0(\Omega) = \left\{ v \in X : \quad \sum_{k=1}^{K} (\mathbf{q} \cdot \mathbf{n}, v)_{\partial\Omega_k} = 0, \quad \mathbf{q} \in H_0(\mathrm{div}, \Omega) \right\}.$$

Now we define the space M of those $\psi = (\psi_1, \cdots, \psi_K)$ with components $\psi_k \in H_*^{-1/2}(\partial\Omega_k)$ such that there weak traces on the boundaries represent a weak trace of a function in $H_0(\mathrm{div}, \Omega)$, i.e.

$$M = \{\psi : \text{ there exists } \mathbf{q} \in H_0(\mathrm{div}, \Omega) \text{ s.t. for } \ k = 1, \cdots, K, \ \psi_k = \mathbf{q} \cdot \mathbf{n}_k\}.$$

The space M is provided with the norm

$$||\psi||_M = \inf\left\{||\mathbf{q}||_{H(\mathrm{div},\Omega)} : \quad \mathbf{q} \in H_0(\mathrm{div}, \Omega), \quad \mathbf{q} \cdot \mathbf{n}_k = \psi_k, \forall k\right\},$$

where $H_*^{-1/2}(\partial\Omega_k)$ is the dual space of $H_*^{1/2}(\partial\Omega_k)$ with $< \cdot, \cdot >_{*,\partial\Omega_k}$ pairing, $H_*^{1/2}(\partial\Omega_k) = H^{1/2}(\partial\Omega_k)$ if $\partial\Omega_k \cap \partial\Omega = \emptyset$ and $H_*^{1/2}(\partial\Omega_k) = H_{00}^{1/2}(\partial\Omega_k \setminus \partial\Omega)$ if $\partial\Omega_k \cap \partial\Omega \neq \emptyset$. Basically speaking the constraints on the distributions $\psi \in M$ imply that the jumps across the interfaces Γ_{kl} vanish.

We now define the bilinear form: $B : X \times M \to R$ by

$$B(v, \phi) = \sum_{k=1}^{K} < v_k, \phi_k >_{*,\partial\Omega_k},$$

so that it follows from Hahn-Banach Theorem that

$$H_0^1(\Omega) = \{v \in X, \quad B(v, \phi) = 0, \quad \phi \in M\}.$$

Similarly, the bilinear form $A : X \times X \to \mathcal{R}$ is defined by

$$A(u, v) = \sum_{k=1}^{K} \int_{\Omega_k} \nabla u_k \cdot \nabla v_k dx.$$

We consider the following model problem: find $u \in H_0^1(\Omega)$ such that

$$A(u, v) = (f, v), \quad v \in H_0^1(\Omega). \tag{1}$$

Its primal hybrid formulation is therefore defined by: find $(u, \psi) \in X \times M$ such that

$$\begin{array}{lcll} A(u, v) + B(v, \psi) & = & (f, v), & v \in X, \\ B(u, \phi) & = & 0, & \phi \in M. \end{array} \tag{2}$$

We have the following equivalent result: Problem (2) has a unique solution $(u, \psi) \in X \times M$, and the first component $u \in H_0^1(\Omega)$ is also the solution of problem (1). Moreover, we have

$$\psi_k = A\nabla u_k \cdot \mathbf{n}_k, \quad k = 1, \cdots, K \quad \text{and} \quad ||u||_{H^1(\Omega)} + ||\psi||_M \leq C||f||_{L^2(\Omega)}.$$

3 Finite Volume Element Formulation

Let the triangulation $\mathcal{T}_{h_k}$ of each subdomain Ω_k, $1 \leq k \leq K$, be such that

$$\overline{\Omega}_k = \cup_{T \in \mathcal{T}_{h_k}} \overline{T}, \quad h_k = \max_{T \in \mathcal{T}_{h_k}} h_T, \quad \text{and} \quad h_T = \sup_{x,y,\in T} d(x, y).$$

For piecewise linear finite element subspaces of $H^1_*(\Omega_k)$ on $\mathcal{T}_{h_k}$, we set

$$X_{\delta,k} = \{v_{\delta,k} \in C(\Omega_k) : \quad v_{\delta,k}|_T \in P_1(T), \quad T \in \mathcal{T}_{h_k}, \quad v_{\delta,k}|_{\partial\Omega \cap \partial\Omega_k} = 0\},$$

and the global finite element spaces

$$X_\delta = \prod_{k=1}^{K} X_{\delta,k}, \quad \text{where} \quad \delta = (h_1, h_2, \cdots, h_K).$$

Notice that the trace of the triangulation $\mathcal{T}_{h_k}$ over Γ_{kl}, $1 \leq k \leq K$, $l \in \overline{k}$, with vertices $v_{1,kl}$ and $v_{2,kl}$ results in a regular triangulation denoted by t_{kl}, where $\overline{k}$ is the class of the indices $l \in \underline{k}$ with $l > k$ and $\underline{k}$ is denotes the set of all indices l so that kl exist. The trace space $W_{\delta,kl}$ of the functions in $X_{\delta,k}$ is given by (see Figure 2):

$$W_{\delta,kl} = \{\phi_{\delta,kl} \in C(\Gamma_{kl}) : \quad t \in t_{kl}, \quad \phi_{\delta,kl} \in P_1(t)\},$$

the approximation of the local Lagrange multiplier is defined as

$$M_{\delta,kl} = \{\phi_{\delta,kl} \in W_{\delta,kl} : \quad t \in t_{kl}, \quad \text{or} \quad \phi_{\delta,kl} \in P_0(t) \quad \text{if} \quad v_{1,kl} \quad or \quad v_{2,kl} \in t\},$$

and the global finite element space on the interface is

$$M_\delta = \prod_{k=1}^{K} \prod_{l \in \overline{k}} M_{\delta,kl}.$$

We now define a bilinear form on $X_\delta \times M_\delta$ by

$$B(v_\delta, \phi_\delta) = \sum_{k=1}^{K} < v_{\delta,k}, \phi_{\delta,k} >_{*,\partial\Omega_k} = \sum_{k=1}^{K} \sum_{l \in \overline{k}} \int_{\Gamma_{kl}} \phi_{\delta,kl}(v_{\delta,k} - v_{\delta,l}) ds.$$

Thus, the mortar finite element approximation of the solution of (2) is defined by (see, e.g. [2, 6, 7]):

$$\begin{array}{lcll} A(u_\delta, v_\delta) + B(v_\delta, \psi_\delta) & = & (f, v_\delta), & v_\delta \in X_\delta, \\ B(u_\delta, \phi_\delta) & = & 0, & \phi_\delta \in M_\delta. \end{array} \tag{3}$$

If the space V_δ of nonconforming approximations of functions in $H^1_0(\Omega)$ is introduced by:

$$V_\delta = \{v_\delta \in X_\delta : \quad B(v_\delta, \phi_\delta) = 0, \quad \phi_\delta \in M_\delta\},$$

then the problem (3) is equivalent to the problem of finding $u_\delta \in V_\delta$ such that

$$A(u_\delta, v_\delta) = (f, v_\delta) \quad v_\delta \in V_\delta. \tag{4}$$

Now we shall introduce the mortar finite volume element approximation of the model problem (3). For a given triangulation $\mathcal{T}_{h_k}$, we construct a dual mesh $\mathcal{T}^*_{h_k}$ based upon $\mathcal{T}_{h_k}$ whose elements are called control volumes.

There are various ways of introducing regular control volume grids $\mathcal{T}_\delta^*$. In the most popular control volume partitions, the medicenter of the finite element T is connected with the midpoints of the edges of T. These types of volumes can be introduced for any finite element partition $\mathcal{T}_{h_k}$ and leads to relatively simple calculations. If the vertex is on the interface Γ_{kl}, then "half" control volume (shaded regions in Figure 1) is used.

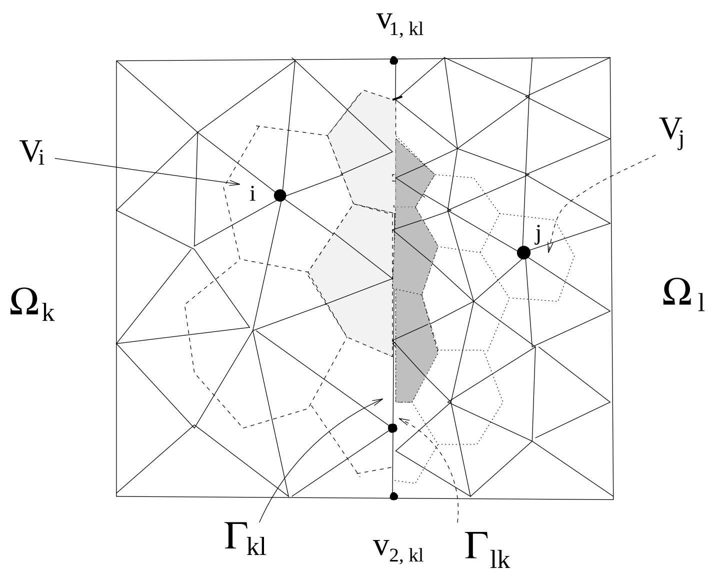

Figure 1: Interfaces Γ_{kl} and Γ_{lk} with $v_{1,kl}$ and $v_{2,kl}$ as two end points, triangulation $\mathcal{T}_{h_k}$ and $\mathcal{T}_{h_l}$, and the volumes in Ω_k and Ω_l. The triangulation t_{kl} and t_{lk} are different on the interface due to non-matching grids.

For the finite element space X_δ we can define its dual volume element space $X_\delta^* = \prod_{k=1}^K X_{\delta,k}^*$, where

$$X_{\delta,k}^* = \{v_k \in L^2(\Omega_k) : \ v_k|_V \ \text{ is constant over } V \in \mathcal{T}_{h_k}^* \ \text{ and } v_k|_{\partial\Omega\setminus\partial\Omega_k} = 0\}.$$

Obviously, $X_{\delta,k}^*$ =span$\{\chi_{i,k}(V) : \ V \in \mathcal{T}_{h_k}^*\}$, where $\chi_{i,k}$ is the characteristic function of the volume $V_{i,k}$. Let $I_{h_k} : C(\Omega_k) \to X_{\delta,k}$ be the interpolation operator and $I_{h_k}^* : C(\Omega_k) \to X_{\delta,k}^*$ be the piecewise constant interpolation operator, that is

$$I_{h_k}^* u = \sum_{x_{i,k} \in N_{h_k}} u_{i,k}\chi_{i,k}(x), \ \text{ where } u_{i,k} = u(x_{i,k}).$$

Then we set $I_\delta = \prod_{k=1}^K I_{h_k}$ and $I_\delta^* = \prod_{k=1}^K I_{h_k}^*$. With the above preparation, we can combine the finite volume approximation (see, e.g. [10, 14, 15, 16]) with the mortar approach to define our *mortar finite volume element method*: find $(u_\delta, \psi_\delta) \in X_\delta \times M_\delta$ such that

$$\begin{array}{lcll} A(u_\delta, I_\delta^* v_\delta) + B(v_\delta, \psi_\delta) & = & (f, I_\delta^* v_\delta), & v_\delta \in X_\delta, \\ B(u_\delta, \phi_\delta) & = & 0, & \phi_\delta \in M_\delta, \end{array} \tag{5}$$

where

$$A(u_\delta, I_\delta^* v_\delta) = -\sum_{k=1}^{K} \sum_{j \in N_{h_k}} v_{j,k} \int_{\partial V_{j,k}} A(x) \nabla u_{\delta,k} \cdot \mathbf{n}_k ds,$$

$$(f, I_\delta^* v_\delta) = \sum_{k=1}^{K} \sum_{j \in N_{h_k}} v_{j,k} \int_{V_{j,k}} f(x) dx.$$

This problem is equivalent to the following problem: find $u_\delta \in V_\delta$ such that

$$A(u_\delta, I_\delta^* v_\delta) = (f, I_\delta^* v_\delta), \quad v_\delta \in V_\delta. \tag{6}$$

Remark: We keep the same piecewise linear element spaces on the interfaces and formulate our mortar finite volume approximations only on the subdomains. This alone in fact is enough to preserve the basic feature of finite volume element method, that is, both (5) and (6) are locally conservative. The weak compatibility condition of the spaces X_δ and M_δ are satisfied automatically:

$$\{\phi_\delta : \quad B(v_\delta, \phi_\delta) = 0, \quad \forall v_\delta \in X_\delta\} = \{0\}, \tag{7}$$

which guarantees that there is no spurious modes generated for the normal derivatives of the solution using this discretization. In other words, our mortar finite volume element formulation has the nice properties of mortar finite element method.

In [13] we have introduced another formulation of the mortar finite volume element method with piecewise constant volume element approximation on the interfaces. The stability, convergence and error estimates for this type of scheme can be obtained in the framework presented above. Similarly, geometrically nonconforming subdomain methods or overlapping domain methods (see Figure 3) can be introduced as well (see [13]).

4 Error Estimates

The following error estimate has been proved in [13]:

Theorem 4.1 *Assume that $\mathcal{T}_\delta$ is regular, then the unique solution pair $(u_\delta, \psi_\delta) \in X_\delta \times M_\delta$ exists for the finite volume element mortar formulation and satisfies the error estimates:*

$$||u - u_\delta||_X \le C \sum_{k=1}^{K} h_k ||u||_H^2(\Omega_k) + C \sum_{k=1}^{K} h_k ||f||_{L^2(\Omega)}.$$

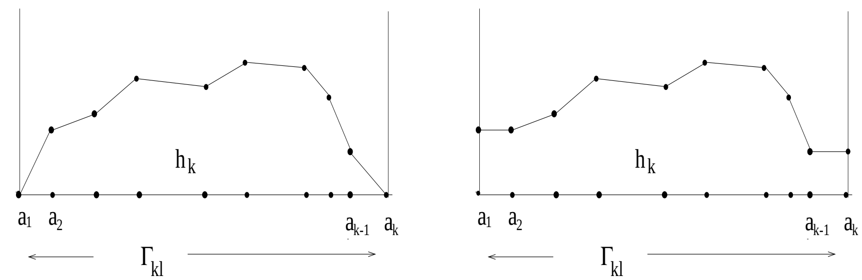

Figure 2: Left: a function from $W_{\delta,kl}$; right: a function from $M_{\delta,kl}$.

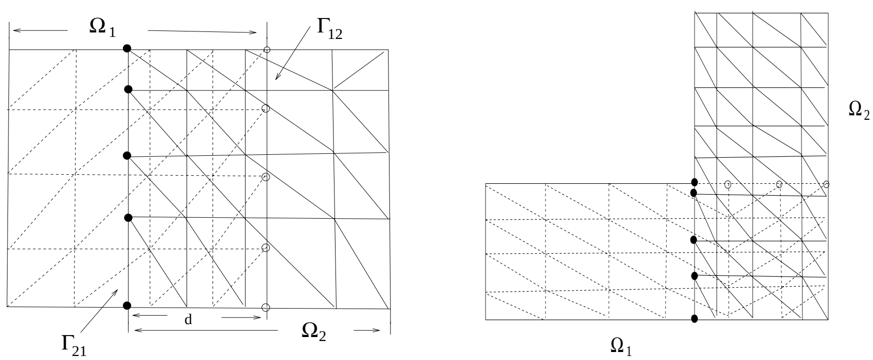

Figure 3: R-shaped with overlapping parameter d and L-shaped domains.

A similar estimate is valid for the M-norm of the error in the Lagrange multipliers as well.

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