FINITE VOLUME ELEMENT APPROXIMATIONS OF INTEGRO-DIFFERENTIAL PARABOLIC PROBLEMS

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In this paper we study finite volume element approximations for two-dimensional parabolic integro-differential equations, arising in modeling of nonlocal reactive flows in porous media. These types of flows are also called NonFickian flows and exhibit mixing length growth. For simplicity we only consider linear finite volume element methods, although higher-order volume elements can be considered as well under this framework. It is proved that the finite element volume approximations derived are convergent with optimal order in H^1- and L^2-norm and superconvergent in a discrete H^1-norm. By examining the relationships between finite volume element and finite element approximations, we prove convergence in L^∞- and $W^{1,\infty}$-norms. These results are also new for finite volume element methods for elliptic and parabolic equations.

1 Introduction

Here we consider finite volume element discretizations of the following initial value problem: Find $u = u(x,t)$ such that

$$
\begin{aligned}
u_t - \nabla\cdot(A\nabla u) - \int_0^t \nabla\cdot(B\nabla u(s))ds = f, \qquad x \in \Omega, \qquad 0 < t \le T,\\
u(x,t) = 0, \qquad x \in \partial\Omega, \qquad 0 < t \le T, \quad (1)\\
u(x,0) = u_0(x), \qquad x \in \Omega,
\end{aligned}
$$

where Ω is a bounded convex polygon in R^2 with a boundary $\partial\Omega$, $A = \{a_{i,j}(x)\}$ is a 2×2 symmetric matrix that is uniformly positive definite in Ω; $B = \{b_{i,j}(x,t,s)\}$ is a 2×2 matrix and $f = f(x,t)$ and $u_0(x)$ are known functions, which are assumed to be smooth and satisfy certain compatibility conditions for $x \in \Omega$ and $t = 0$ so that Eq. (1) has a unique solution in a certain Sobolev space.

This model appears in transport of contaminants in aquifers, an area of active interdisciplinary research of mathematicians, engineers and life scientists (see, e.g. [5,6]). From a mathematical point of view, the evolution of either a passive or reactive chemical within a velocity field exhibiting many scales defies representation using classical Fickian theory. The evolution of a chemical in such a velocity field when modeled by Fickian type theories leads to a dispersion tensor whose magnitude depends upon the time-scales of observation. In order to avoid such difficulties, a new class of nonlocal models of transport has been derived. In this case, the constitutive relations involve either integrals or higher-order derivatives, which take multi-scales into consideration. We refer the reader to [5] for derivations of the mathematical models and for the precise hypotheses and analyses.

Mathematical formulations of this kind arise naturally also in various engineering applications, such as heat conduction, radioactive nuclear decay in fluid flows, [17] non-Newtonian fluid flows, viscoelastic deformations of materials with memory, [15] biotechnology etc. One very important characteristic of all these models is that they all express conservation of a certain quantity (mass, momentum, heat, etc.) for any subdomain. This in many applications is the most desirable feature of the approximation method when it comes to numerical solution of the corresponding initial boundary value problem.

Problems like Eq. (1) have been extensively treated by finite element, finite difference, and collocation methods (see, e.g. [3,11,16]), while very few results are known for finite volume methods. The finite element method is approximately locally conservative and therefore in the asymptotic limit (i.e. when the grid step-size tends to zero) it will produce adequate results. However, this could be a disadvantage when relatively coarse grids are used. The finite volume method exactly conserves the flux (heat, mass, etc) over each computational cell. This important property, combined with its adequate accuracy and ease of the implementation, has contributed to the recently renewed interest in the method.

The finite volume element method discretization technique can be characterized as an approximation in the framework of the standard Petrov-Galerkin weak formulation. It involves two spaces: the solution space S_h of piece-wise linear continuous functions over the finite element partition, and the test space S_h^* of piece-wise constant functions over the finite (control) volume partition. The test space S_h^* essentially ensures the local conservation property of the method similar to that of the mixed finite elements. However, in contrast to the mixed method it leads to definite but, in general, nonsymmetric problems.

To the best of the authors' knowledge, the finite volume element approximations of the Eq. (1) have not been studied before. We first introduce the concept of finite volume element approximations, the domain partitioning into finite elements and finite (control) volumes, various discrete norms and notations, and then we state some auxiliary results. First, we characterize the finite dimensional spaces S_h and S_h^* and show the weak coercivity and the boundness of the corresponding bilinear form on $S_h \times S_h^*$. Once these fundamentals have been established, we derive our semi-discrete and fully-discrete (in time) locally conservative discretization schemes. Our main goal is to analyze the convergence rates of these schemes in H^1- and L^2-norms under minimal regularity of the solution. Namely, we obtain optimal order first-order error estimates in the H^1-norm for solutions in $H^2(\Omega)$ and second-order estimates in the L^2-norm under the additional assumption that the solution u is in $W^{3,p}(\Omega)$ for $1 < p < 2$. This indicates that in terms of regularity, the L^2-estimate is sub-optimal. Further results concerning L^∞-error estimates and superconvergence can be found in our paper. [8]

2 Finite Volume Element Approximation

In this section, we introduce all notations that are necessary for the further consideration and derive the finite volume element discretization of the model problem. Next, we state some auxiliary results, introduce a Ritz-Volterra projection, and study its properties. The complete proofs of the corresponding lemmas and theorems can be found in. [8]

We use the standard notations for Sobolev spaces $W^{s,p}(\Omega)$ for $1 \le p \le \infty$ for functions having generalized derivatives of order s, integrable with power p in Ω. The norm $W^{s,p}(\Omega)$ is defined by

$$||u||_{s,p,\Omega} = ||u||_{s,p} = \left(\int_\Omega \sum_{|\alpha| \le s} |D^\alpha u|^p dx \right)^{1/p}, \text{ for } 1 \le p < \infty,$$

with the standard modification for $p = \infty$. In order to simplify the notations, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$, and we skip the index $p = 2$ and Ω when possible, i.e. $||u||_{s,2,\Omega} = ||u||_{s,\Omega} = ||u||_s$. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$.

For functions defined on the cylinder $\Omega \times J$, where $J \equiv [0, T]$, we shall also use the notation of spaces of functions with finite norms. Namely, $L^p(X)$

will denote the Banach space of functions equipped with the norm:

$$\left(\int_0^T ||u||_X^p dt \right)^{1/p}, \qquad 1 \le p < \infty.$$

The domain Ω is split into triangular finite elements K. The elements K are considered to be closed sets and the triangulation is denoted by T_h. Then $\overline{\Omega} = \cup_{K \in T_h} K$ and N_h denotes all nodes (vertices):

$$N_h = \{p : \;\; p \;\text{ is a vertex of element }\; K \in T_h \;\text{ and }\; p \in \bar{\Omega}\}.$$

In order to accommodate the Dirichlet boundary conditions we shall also need the set of vertices internal to Ω, denoted by N_h^0, i.e. $N_h^0 = N_h \cap \Omega$. For a given vertex x_i, we define by $\Pi(i)$ the index set of all neighbors of x_i in N_h^0.

For a given triangulation T_h, we construct a dual mesh T_h^* based upon T_h, whose elements are called control volumes. In the finite volume methods there are various ways to introduce the control volumes. Almost all approaches can be described in the following general scheme: in each triangle $K \in T_h$ a point q is selected; similarly on each of the three edges $\overline{x_i x_j}$ of K a point x_{ij} is selected; then q is connected with the points x_{ij} by straight lines γ_{ij}. Thus, around each vertex $x_j \in N_h^0$, we associate the control volume $V_j \in T_h^*$, which consists of the union of the sub-elements $K \in T_h$, which has x_j as a vertex. Also let γ_{ij} denote the interface of two control volumes V_i and V_j: $\gamma_{ij} = V_j \cap V_j$, $j \in \Pi(i)$ (see Figure 1 and 2).

We call the partition T_h^* *regular or quasiuniform* if there exists a positive constant $C > 0$, independent of h, such that

$$C^{-1} h^2 \le \text{meas}(V_i) \le Ch^2, \qquad \text{for all} \qquad V_i \in T_h^*.$$

Here h is the maximal diameter of all elements $K \in T_h$. In this paper we shall deal with a regular triangulation T_h^*.

The partition T_h^* is said to be *symmetric* if $x_{ij} = \gamma_{ij} \cap \overline{x_i x_j}$ is the middle point of the line segment $\overline{x_i x_j}$, and x_{ij} is the middle point of γ_{ij} or γ_{ij} has two perpendicular axes of symmetry and x_{ij} is their intersection point.

There are various ways of introducing regular control volume grids T_h^*. The following two partitions are widely used in the finite volume element method; we shall use in our paper (see Figures 2 and 2.

In the first control volume partition, the point q is chosen to be the medicenter (the center of gravity) of the finite element K and the points x_{ij} are chosen to be the midpoints of the edges of K (see Figure 2). This type of control volume can be introduced for any finite element partition T_h and leads to relatively simple calculations for both 2- and 3-D problems. Besides,

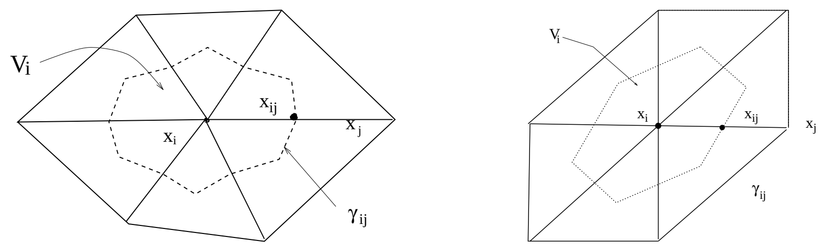

Figure 1. Volumes with medicenters as internal points and interface γ_{ij} of V_i and V_j.

if the finite element partition T_h is locally regular, i.e. there is a constant C such that $Ch_K^2 \leq \text{meas}(K) \leq h_K^2, \quad \text{diam}(K) = h_K$ for all elements $K \in T_h$, then the finite volume partition T_h^* is also locally regular.

In the second type of control volume, the point q is the circumcenter of the element K, i.e. the center of the circumscribed circle of K and x_{ij} are the midpoints of the edges of K. This type of control volume forms the so-called Voronoi meshes. Then obviously γ_{ij} are the perpendicular bisectors of the three edges of K (see Figure 2) This construction requires that all finite elements are triangles of acute type, which we shall assume whenever such a triangulation is used.

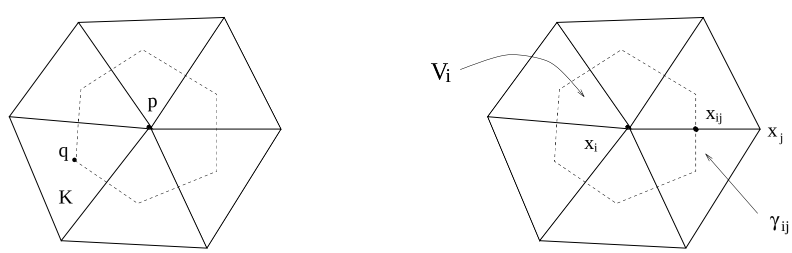

Figure 2. Volumes with circumcenters as internal points and interface γ_{ij} of V_i and V_j.

We are now ready to define the finite element space S_h of linear elements:

$$S_h = \{v \in C(\Omega): \quad v|_K \quad \text{is linear for all} \quad K \in T_h \quad \text{and } v|_{\partial\Omega} = 0\},$$

and its dual volume element space S_h^*:

$$S_h^* = \{v \in L^2(\Omega): \quad v|_V \quad \text{is constant for all} \quad V \in T_h^* \quad \text{and } v|_{\partial\Omega} = 0\}.$$

Obviously, $S_h = \text{span}\{\phi_i(x) : x_i \in N_h^0\}$ and $S_h^* = \text{span}\{\chi_i(x) : x_i \in N_h^0\}$, where ϕ_i are the standard nodal linear basis functions associated with the node x_i and χ_i are the characteristic functions of the volume V_i. Let $I_h : C(\Omega) \to S_h$ be the interpolation operator and $I_h^* : C(\Omega) \to S_h^*$ constant interpolation operators, respectively. That is

$$I_h u = \sum_{x_i \in N_h} u(x_i)\phi_i(x), \qquad \text{and} \qquad I_h^* u = \sum_{x_i \in N_h} u(x_i)\chi_i(x).$$

The semi-discrete finite volume element approximation u_h of (1) is a solution to the problem: find $u_h(t) \in S_h$ for $t > 0$ such that

$$(u_{h,t}, v_h) + A(u_h, v_h) + \int_0^t B(t, s; u_h(s), v_h)ds = (f, v_h), \qquad v_h \in S_h^* \tag{2}$$
$$u_h(0) = u_{0,h} \in S_h,$$

or

$$\begin{aligned} &(u_{h,t}, I_h^* v_h) + A(u_h, I_h^* v_h) \\ &\qquad + \int_0^t B(t, s; u_h(s), I_h^* v_h)ds = (f, I_h^* v_h), \qquad v_h \in S_h. \end{aligned} \tag{3}$$

Here the bilinear form $A(u, v)$ is defined by

$$A(u, v) = \begin{cases} -\displaystyle\sum_{x_i \in N_h} v_i \int_{\partial V_i} A\nabla u \cdot \mathbf{n} dS_x, & (u, v) \in H_0^1 \cap H^2 \times S_h^*, \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v dx, & (u, v) \in H_0^1 \times H_0^1, \end{cases} \tag{4}$$

where $\mathbf{n}$ denotes the outer-normal direction to the domain under consideration. The form $B(\cdot, \cdot)$ is defined in a similar way.

Remark 1 *We use the same notation for the bilinear forms A and B defined in two different ways on the pair of spaces $H_0^1 \times H_0^1$ and $H_0^1 \cap H^2 \times S_h^*$, correspondingly. We hope that this will not lead to serious confusion while it simplifies tremendously the notations and the overall exposition of the material.*

Next, we define the fully-discrete time stepping scheme. Let $\Delta t > 0$ be a time-step size and $t_n = n\Delta t$, $n = 0, 1, \cdots$, and $g^n = g(t_n)$.

The backward Euler scheme is defined to be the solution of $u_h^n \in S_h$ such that

$$\left(0\frac{u_h^n - u_h^{n-1}}{\Delta t}, I_h^* v_h\right) + A(u_h^n, I_h^* v_h) + \sum_{k=0}^{n-1} \omega_{n,k} B(t_n, t_k; u_h^k, I_h^* v_h) = (f^n, I_h^* v_h),$$
$$u_h^0(0) = u_{0,h} \in S_h, \tag{5}$$

where $\omega_{n,j}$ are the weights and the quadrature formula for evaluating the integral in time. We assume that the error of the quadrature formula satisfies the estimate:

$$\left| \int_0^{t_n} M(t_n, s) g(s) ds - \sum_{j=1}^{n-1} \omega_{n,j} M(t_n, t_j) g(t_j) \right| \leq C \Delta t \int_0^{t_n} (|g| + |g'|) dt.$$

We next define some discrete norms on S_h and S_h^*, which are used in our analysis:

$$|u_h|_{0,h}^2 = (u_h, u_h)_{0,h}, \ \text{ with } \ (u_h, v_h)_{0,h} = \sum_{x_i \in N_h} \text{meas}(V_i) u_i v_i = (I_h^* u_h, I_h^* v_h),$$

$$|u_h|_{1,h} = \sum_{x_i \in N_h} \sum_{x_j \in \Pi(i)} \text{meas}(V_i) \left((u_i - u_j)/d_{ij} \right)^2,$$

$$||u_h||_{1,h}^2 = |u_h|_{0,h}^2 + |u_h|_{1,h}^2, \qquad |||u_h|||_0^2 = (u_h, I_h^* u_h),$$

where $d_{ij} = d(x_i, x_j)$, the distance between x_i and x_j.

In the lemmas below, we assume that the matrix $A(x)$ may have jumps, which are aligned with the finite element partition T_h and over each element the entries of the matrix $A(x)$ are C^1-functions. We also assume that T_h is a regular partition of Ω.

Lemma 1 *(See, e.g.* [2,12]*) There exist two positive constants $C_0, C_1 > 0$, independent of h, such that*

$$\begin{aligned} C_0 |v_h|_{0,h} \leq ||v_h||_0 \leq C_1 |v_h|_{0,h}, &\qquad v_h \in S_h, \\ C_0 |||v_h|||_0 \leq ||v_h||_0 \leq C_1 |||v_h|||_0, &\qquad v_h \in S_h, \\ C_0 ||v_h||_{1,h} \leq ||v_h||_1 \leq C_1 ||v_h||_{1,h}, &\qquad v_h \in S_h. \end{aligned}$$

Lemma 2 *(See, e.g.* [2,12]*) There exist two positive constants $C_0, C_1 > 0$, independent of h, and $h_0 > 0$ such that for all $0 < h \leq h_0$*

$$|A(u_h, I_h^* v_h)| \leq C_1 |u_h|_{1,h} |v_h|_{1,h}, \qquad u_h, \ v_h \in S_h, \tag{6}$$

$$A(u_h, I_h^* u_h) \geq C_0 |u_h|_{1,h}^2, \qquad u_h, \ v_h \in S_h. \tag{7}$$

Lemma 3 *(See, e.g.* [2,8,12]*) If T_h is regular, then there exists a positive constant $C > 0$, independent of h, such that*

$$|A(u - I_h u, I_h^* v_h)| \leq Ch ||u||_2 |v_h|_{1,h}, \qquad v_h \in S_h. \tag{8}$$

In the case of symmetric partitions and smooth solutions and due to the cancellation in the local truncation error, one can get higher-order approximations by the same finite elements. Namely, we can prove:

Lemma 4 *(see, e.g. [8]) If T_h is regular and symmetric, then there exists a positive constant $C > 0$, independent of h, such that*

$$|A(u - I_h u, I_h^* v_h)| \leq Ch^2 ||u||_3 |v_h|_{1,h}, \quad v_h \in S_h. \tag{9}$$

In fact, if the triangulation T_h is regular and any two adjacent elements form an approximate parallelogram, then it can be proved that [12]:

$$|A(u - I_h u, I_h^* v_h)| \leq Ch^2 \left(||u||_3 + ||u||_{2,\infty}\right) |v_h|_{1,h}, \quad v_h \in S_h.$$

This means that almost symmetric grids have the same convergence rates (for smooth solutions) as the symmetric ones.

For any fixed $0 < t \leq J$, one can define the Ritz projection $R_h u$ of function $u(x,t)$ where the operator $R_h : H_0^1 \cap H^2 \to S_h$ so that

$$A(u - R_h u, I_h^* v_h) = 0, \quad \text{for all} \quad v_h \in S_h. \tag{10}$$

Remark 2 *The results of the above lemmas will lead to the following results: (a) if the partition T_h is regular (quasiuniform) and u is H^2-regular, then*

$$||u - R_h u||_1 \leq Ch||u||_2;$$

(b) if the partition is regular and symmetric and u is H^3-regular, then

$$||u - R_h u||_{1,h} \leq Ch^2 ||u||_3.$$

The estimates stated above for the Ritz-projection are very useful in the analysis of finite element and finite volume methods for parabolic equations. However, these estimates will produce a suboptimal error estimate for the discrete schemes for integro-differential equation. In order to obtain optimal order estimates, we need a projection which also takes into account the integral term. This type of projection has been called by Cannon and Lin[3] the Ritz-Volterra projection and has been used in the context of the finite element method.

Now we define the Ritz-Volterra projection $V_h u$ of a function u defined on the cylinder $\Omega \times J$ and state its approximation properties. The full proofs of the theorems stated below can be found, for example, in Ewing et. al.[8].

The Ritz-Volterra projection $V_h : L^\infty(H_0^1 \cap H^2) \to L^\infty(S_h)$ is defined for $0 \leq t \leq T$ by

$$A(u - V_h u, I_h^* v_h) + \int_0^t B(t, s; u(s) - V_h u(s), I_h^* v_h) ds = 0, \quad v_h \in S_h. \tag{11}$$

Theorem 1 *Assume that the mesh T_h^* is regular and (1) $D_t^l u \in L^\infty(H^2)$ for all $0 \leq l \leq k$ for some integer $k \geq 0$. Then, the Ritz-Volterra projection*

$V_h u$ is well-defined and for any $t > 0$ and $0 \le l \le k$, there is a constant $C = C(t) > 0$, independent of h, such that the following estimate holds true:

$$||D_t^l(u - V_h u)||_1 \le Ch \left(\sum_{j=0}^{l} ||D_t^j u||_2 + \int_0^t \sum_{j=0}^{l} ||D_t^j u(s)||_2 ds \right). \qquad (12)$$

In addition, if T_h^* is also symmetric and $D_t^l u \in L^\infty(H^3)$, then we have

$$|D_t^l(I_h u - V_h u)|_1 \le Ch^2 \left(\sum_{j=0}^{l} ||D_t^j u||_3 + \int_0^t \sum_{j=0}^{l} ||D_t^j u(s)||_3 ds \right). \qquad (13)$$

Now we consider an estimate in the L^2-norm for the Ritz-Volterra projection that is optimal with respect to the order of convergence but requires $W^{3,p}$-regularity of the solution. Therefore, this estimate is suboptimal with respect to the regularity of the solution and makes sense for p close to 1. Namely, we prove the following result:

Theorem 2 *Assume that the partition T_h is regular and for some $p > 1$ and an integer $k \ge 0$, $D_t^l u \in L^\infty(W^{3,p}(\Omega))$ for $0 \le l \le k$. Then for each $t > 0$ there exists a positive constant $C = C(t) > 0$, independent of h, such that for $0 \le l \le k$*

$$||D_t^l(u - V_h u)|| \le Ch^2 \sum_{j=0}^{l} \left(||D_t^j u||_{3,p} + \int_0^t ||D_t^j u||_{3,p} ds \right). \qquad (14)$$

3 Error Estimates in L^2- and H^1-norms

In this section, we prove error estimates for the finite volume element approximation in L^2- and H^1-norms.

Theorem 3 *Assume that T_h is regular and $u, D_t u \in L^\infty(H_0^1 \cap W^{3,p})$, for some $p > 1$ and for all $t \ge 0$. Assume also that the approximation $u_h(0)$ of the initial data satisfies $||u_h(0) - u_0|| \le Ch^2 ||u_0||_2$. Then there exists a constant, independent of h and u such that for all $t \ge 0$*

$$||u - u_h|| \le Ch^2 \left(||u_0||_{3,p} + \int_0^t ||u_t||_{3,p} ds \right). \qquad (15)$$

Proof: Note that the interesting case is when p is close to 1. Let $u - u_h = (u - V_h u) + (V_h u - u_h) = \rho + \theta$, where V_h is the Ritz-Volterra projection defined above, so that by Theorem 2 we have

$$||\rho(t)|| \leq Ch^2 \left(||u||_{3,p} + \int_0^t ||u||_{3,p} ds \right), \tag{16}$$

$$||\rho_t(t)|| \leq Ch^2 \left(||u||_{3,p} + ||u_t||_{3,p} + \int_0^t ||u||_{3,p} ds \right). \tag{17}$$

Besides, by Eq. (3) and Eq. (11), θ satisfies the equation

$$\begin{aligned} &(\theta_t, I_h^* v_h) + A(\theta, I_h^* v_h) \\ &\qquad + \int_0^t B(t,s;\theta(s), I_h^* v_h) ds = -(\rho_t, I_h^* v_h), \qquad v_h \in S_h. \end{aligned} \tag{18}$$

Set $v_h = \theta \in S_h$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|||\theta|||_0^2 + C_0||\theta||_1^2 &\leq C \int_0^t ||\theta||_1 ds\, ||\theta||_1 + ||\rho_t||\, ||\theta|| \\ &\leq \frac{c_0}{2}||\theta||^2 + C\int_0^t ||\theta||_1^2 ds + ||\rho_t||\, ||\theta||, \end{aligned}$$

so that after integration in time from 0 to t, we get

$$||\theta||^2 + \int_0^t ||\theta||_1^2 ds \leq C \left(||\theta(0)||^2 + \int_0^t ||\rho_t||\, ||\theta|| ds + \int_0^t \int_0^t \int_0^\tau ||\theta(\tau)||_1^2 d\tau ds \right).$$

Thus, Gronwall's inequality leads to

$$||\theta||^2 + \int_0^t ||\theta||_1^2 ds \leq C||\theta(0)||^2 + \frac{1}{2} \sup_{0<s<t} ||\theta(s)||^2 + C\left(\int_0^t ||\rho_t|| ds \right)^2,$$

and therefore

$$||\theta|| \leq C\left(||\theta(0)|| + C\int_0^t ||\rho_t|| ds \right) \leq Ch^2 \left(||u_0||_{3,p} + \int_0^t ||u_t||_{3,p} ds \right).$$

Hence, Theorem 3 follows from (16), (17), and the above inequality together with triangle inequality.

Now we derive the error estimate for the discrete H^1-norm, which can be interpreted as superconvergence of the gradient of the solution at some particular points.

Theorem 4 *Assume that T_h^* is regular and symmetric and $u, u_t \in L^\infty(H_0^1 \cap H^3)$, for all $t \geq 0$. Assume also that the approximation $u_h(0)$ of the initial* *data satisfies* $||u_h(0) - u_0|| \le Ch^2||u_0||_2$. *Then for any* $t > 0$ *there exists a constant* $C = C(t) > 0$, *independent of* h *and* u, *such that*

$$|u - u_h|_{1,h} \equiv |I_h u - u_h|_1 \le Ch^2 \left(||u_0||_3 + \int_0^t ||u_t||_3 ds \right). \tag{19}$$

Proof: We first note that $|u-u_h|_{1,h} \le |I_h u - V_h u|_{1,h} + |V_h u - u_h|_{1,h}$. Obviously $|I_h u - V_h u|_{1,h} = |I_h u - V_h u|_1$ and this term has already been estimated in Theorem 1 for $l = 0$:

$$|I_h u - V_h u|_1 \le Ch^2 \left(||u||_3 + \int_0^t ||u(s)||_3 ds \right). \tag{20}$$

Therefore, we need to show that

$$|V_h u - u_h|_1 \equiv |V_h u - u_h|_{1,h} \le Ch^2 \left(||u_0||_3 + \int_0^t ||u_t||_3 ds \right).$$

Set $v_h = \theta_t$ in (18), to get

$$\begin{aligned} |||\theta_t|||_0^1 + \frac{1}{2}\frac{d}{dt}A(\theta, I_h^*\theta_t) &= -(\rho_t, I_h^*\theta_t) - \int_0^t B(t,s;\theta(s), I_h^*\theta_t(t))ds \qquad (21)\\ &\le \frac{1}{2}||\rho_t||^2 + \frac{1}{2}|||\theta|||_0^2 - \frac{d}{dt}\int_0^t B(t,s;\theta(s), I_h^*\theta(t))ds \\ &\quad + B(t,t;\theta(t), I_h^*\theta(t)) + \int_0^t B_t(t,s;\theta(s), I_h^*\theta(t))ds, \end{aligned}$$

where $B_t(t,s;\cdot,\cdot)$ is the time derivative of $B(t,s;\cdot,\cdot)$. This will lead to the following inequality

$$\begin{aligned} |||\theta_t|||_0^1 + \frac{d}{dt}A(\theta, I_h^*\theta_t) &\le ||\rho_t||^2 - \frac{d}{dt}\int_0^t B(t,s;\theta(s), I_h^*\theta(t))ds \qquad (22)\\ &\quad + C|\theta|_1 + C\int_0^t |\theta(s)|_1^2 ds. \end{aligned}$$

By integration and use of the coercivity of $A(\theta, I_h^*\theta_t)$, we get

$$\begin{aligned} \int_0^t ||\theta_t||^2 ds + C_0|\theta|_1^2 &\le C\left(||\theta(0)||_1^2 + \int_0^t ||\rho_t||^2 ds \right) \\ &\quad + \int_0^t B(t,s;\theta(s), I_h^*\theta(t))ds + C\int_0^t |\theta|_1^2 ds \\ &\le C\left(||\theta(0)||_1^2 + \int_0^t ||\rho_t||^2 ds \right) + \frac{C_0}{2}|\theta|_1^2 + C\int_0^t |\theta|_1^2 ds, \end{aligned}$$

and by Gronwall's inequality, we see that

$$\int_0^t ||\theta_t||^2 ds + ||\theta||_1^2 \leq C \left(||\theta(0)||_1^2 + \int_0^t ||\rho_t||^2 ds \right).$$

Noticing that

$$||\theta_0||_1 \leq ||V_h u_0 - u_0||_1 + ||u_0 - u_h(0)||_1 \leq Ch^2 ||u_0||_3,$$

Theorem 2 will imply

$$||\theta||_1^2 \leq Ch^2 \left(||u_0||_3 + \int_0^t ||u_t||_3^2 ds \right)$$

and the required estimate (19) follows from (20).

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