Finite Volume Element Approximations of Nonlocal Reactive Flows in Porous Media

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In this paper we study finite volume element approximations for two-dimensional parabolic integro-differential equations, arising in modeling of nonlocal reactive flows in porous media. These type of flows are also called NonFickian flows with mixing length growth. For simplicity we only consider linear finite volume element methods, although higher order volume elements can be considered as well under this framework. It is proved that the derived finite element volume approximations are convergent with optimal order in H^1- and L^2-norm, and superconvergent in a discrete H^1-norm. By examining the relationships between finite volume element and finite element approximations, we prove convergence in L^∞- and $W^{1,\infty}$-norms. These results are new also for finite volume element methods for elliptic and parabolic equations. © 0000 John Wiley & Sons, Inc.

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I. INTRODUCTION

In this paper we consider finite volume element discretizations of the following initial value problem for the operator equation for $u = u(t)$:

$$u_t + \mathcal{A}u + \int_0^t \mathcal{B}(t,s)u(s)ds = f(t), \quad u(0) = u_0, \tag{1.1}$$

where $\mathcal{A}$ is strongly elliptic differential operator and $\mathcal{B}$ is a second order elliptic differential operator in space. The operators $\mathcal{A}$ and $\mathcal{B}$ incorporate Dirichlet and Neumann

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boundary conditions. The problem (1.1) is an abstract form of an initial boundary value problem for a parabolic integro-differential equation.

This model is very important in transport of reactive and passive contaminates in aquifers, an area of active interdisciplinary research of mathematicians, engineers and life scientists. From a mathematical point of view, the evolution of either a passive or reactive chemical within a velocity field exhibiting on many scales defies representation using classical Fickian theory. The evolution of a chemical in such a velocity field when modeled by Fickian type theories leads to a dispersion tensor whose magnitude depends upon the time-scales of observation. In order to avoid such difficulty a new class of nonlocal models of transport have been derived. In this case, the constitutive relations involve with either integrals or higher order derivatives which take multi-scales into consideration. We refer the reader to [6] and [8] for deriving the mathematical models and for the precise hypothesis and analysis.

Mathematical formulations of this kind arise naturally also in various engineering models, such as nonlocal reactive transport in underground water flows in porous media [7] and [9], heat conduction, radioactive nuclear decay in fluid flows [20], non-Newtonian fluid flows, or viscoelastic deformations of materials with memory (in particular polymers) [19], semi-conductor modeling [1], and biotechnology. One very important characteristic of all these models is that they all express a conservation of a certain quantity (mass, momentum, heat, etc.) in any moment for any subdomain. This in many applications is the most desirable feature of the approximation method when it comes to numerical solution of the corresponding initial boundary value problem.

This type of equations have been extensively treated by finite element, finite difference, and collocation methods in the last ten year [4, 14, 15, 21, 26], while very little results are known for finite volume method. The finite element method conserves the flux approximately and therefore in the asymptotic limit (i.e. when the grid step-size tends to zero) it will produce adequate results. However, this could be a disadvantage when relatively coarse grids are used. Perhaps, the most important property of the finite volume method is that it conserves exactly the flux (heat, mass, etc) over each computational cell. This important property combined with its broad application, adequate accuracy and easiness in the implementation had contributed to the recent renewed interest in the method.

The discretization technique of the finite volume element method can be characterized as an approximation in the framework of the standard Petrov-Galerkin weak formulation. It involves two spaces: the solution space S_h of piece-wise linear over the finite element partition continuous functions, and the test space S_h^* of piece-wise constant functions over the finite (control) volume partition. The test space S_h^* essentially ensures the local conservation property of the method. In this respect, the finite volume element method has the conservation property of the mixed finite elements. However, in contrast to the mixed method it leads to definite but, in general, nonsymmetric problems.

To the best of the author's knowledge, the finite volume element approximations of the problem (1.1) have not been studied before. In this paper, we first introduce the concept of finite volume element approximations, the domain portioning into finite elements and finite (control) volumes, various discrete norms, notations, and state and derive some auxiliary helpful results. The main efforts have been directed to characterize the finite dimensional spaces S_h and S_h^* and to show the weak coercivity and the boundness of the corresponding weak formulations of the bilinear form (associated with the operator $\mathcal{A}$)

on $S_h \times S_h^*$. Once these fundamentals have been established, we derive and study several semi-discrete and fully discrete (in time) schemes.

Our main goal is to analyze the convergence rates of the discretization schemes in various norms under various conditions on the regularity of the solution. The main results of the paper can be summarized in the following way. First we introduce the finite volume element Ritz-Volterra projection, borrowing this concept from [4], and study its properties. In §2 we derive L^2- and H^1- error estimates for the Ritz-Volterra projection and obtain optimal in H^1-norm convergence and super-convergence estimates for the projection. These estimates play fundamental role in deriving optimal rates for the error of the finite volume element method. Namely, we obtain optimal second order estimate in L^2-norm under the additional assumption that the solution u is in $W^{3,p}(\Omega)$ for $1 < p < 2$. This indicates that in term of regularity the estimate is sub-optimal.

Next, we study the error of the Ritz-Volterra projection in L^∞- and $W^{1,\infty}$-norms by exploiting the concept of regularized Green function and using its properties known from the finite element method (see, e.g. [15, 18, 27]). These estimates are used to obtain optimal order error estimates and superconvergence of the gradient of the approximate solution. The regularity required for L^∞ norm estimates is $W^{3,p}$ with $p > 2$, which is worse than $W^{2,\infty}$ but better than $W^{3,\infty}$. The trade off for this higher regularity is that there is no logarithmic factor in L^∞-norm estimates. The superconvergence in $W^{1,\infty}$-norm contains a logarithmic factor.

II. FINITE VOLUME ELEMENT METHOD

In this section we first formulate the mathematical problem and introduce all necessary for the further consideration notation. Next we derive the finite volume element discretization of the model problem, obtain some auxiliary results, introduce Ritz-Volterra projection and study its properties.

A. Problem Formulations and Notations

In this paper we consider the following initial boundary value problem: find $u = u(x,t)$ such that

$$\begin{aligned} u_t - \nabla\cdot(A\nabla u) - \int_0^t \nabla\cdot(B\nabla u(s))ds &= f, \quad x\in\Omega, \quad 0<t\le T, \\ u(x,t) &= 0, \quad x\in\partial\Omega, \quad 0<t\le T, \\ u(x,0) &= u_0(x), \quad x\in\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded convex polygon in R^2 with a boundary $\partial\Omega$, $A = \{a_{i,j}(x)\}$ is a 2×2 symmetric and uniformly in Ω positive definite matrix, $B = \{b_{i,j}(x,t,s)\}$ is 2×2 matrix, and $f = f(x,t)$ and $u_0(x)$ are known functions which are assumed to be smooth so that problem (2.0) has a unique solution in a certain Sobolev space.

Remark 2.1. *One can add to the differential operator $\nabla\cdot(A\nabla u)$ a convection term of the type $\nabla\cdot(\mathbf{b}\, u)$ with $\mathbf{b} = (b_1, b_2)^T$ given vector. Most of the analysis and the constructions derived in this paper will be valid in this case also, provided that the convective term is treated in a right way (see, some details for such approximations of elliptic problems in [16] and [17]).*

Remark 2.2. *One may consider also Neumann and Robin boundary conditions on the whole or on a part of the boundary $\partial\Omega$. The construction of the finite volume approximation and its analysis can be carried out with no additional difficulties. In fact, the finite volume element method was introduced by Baliga and Patankar in [2] as an attempt to approximate the flux boundary conditions by finite differences in a consistent and systematic way.*

We use the standard notations for Sobolev spaces $W^{s,p}(\Omega)$ for $1 \le p \le \infty$ of functions having generalized derivatives of order s, integrable with power p in Ω. The norm is $W^{s,p}(\Omega)$ is defined by

$$||u||_{s,p,\Omega} = ||u||_{s,p} = \left(\int_\Omega \sum_{|\alpha| \le s} |D^\alpha u|^p dx \right)^{1/p}, \text{ for } 1 \le p < \infty$$

with the standard modification for $p = \infty$. In order to simplify the notations we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and skip the index $p = 2$ and Ω when possible, i.e. $||u||_{s,2,\Omega} = ||u||_{s,\Omega} = ||u||_s$. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$. Finally, $H^{-1}(\Omega)$ denotes the space of all bounded linear functionals on $H_0^1(\Omega)$. For a function $u \in H_0^1(\Omega)$ the functional $f \in H^{-1}(\Omega)$ is defined by the inner product (f, u) representing the duality pairing in $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

For functions defined on the cylinder $\Omega \times J$, where $J \equiv [0, T]$, we shall also use the notation of spaces of functions with finite norms. Namely, $L^p(X)$ will denote the Banach space of functions equipped with the norm:

$$\left(\int_0^T ||u||_X^p dt \right)^{1/p}, \qquad 1 \le p < \infty.$$

The problem (2.0) can be written in the form (1.1). First, introduce the operators $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $\mathcal{B} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by the identities

$$(\mathcal{A}u, v) = \int_\Omega A\nabla u \cdot \nabla v dx, \qquad (\mathcal{B}u, v) = \int_\Omega B(t,s)\nabla u \cdot \nabla v dx$$

for any $t, s \in (0, T)$. With some abuse of the notations $(\cdot, \cdot)$ denotes both the $L^2(\Omega)$-inner product and the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

B. Finite Volume Element Approximation

We assume that Ω is a convex polygonal domain. The domain Ω is split into triangular finite elements K. The elements K are considered to be closed sets and the triangulation is denoted by T_h. Then $\overline{\Omega} = \cup_{K \in T_h} K$ and N_h denotes all nodes or vertices:

$$N_h = \{p : \ \ p \ \text{ is a vertex of element } \ K \in T_h \ \text{ and } \ p \in \bar{\Omega}\}.$$

In order to accommodate the Dirichlet boundary conditions we shall also need the set of internal to Ω vertices, denoted by N_h^0, i.e. $N_h^0 = N_h \cap \Omega$. For a given vertex x_i we define by $\Pi(i)$ the index set of all neighbors of x_i in N_h^0.

For a given triangulation T_h, we construct a dual mesh T_h^* based upon T_h which elements are called control volumes. In the finite volume methods there are various ways to introduce the control volumes. Almost all approaches can be described in the following general scheme: in each triangle $K \in T_h$ a point q is selected; similarly on each of the

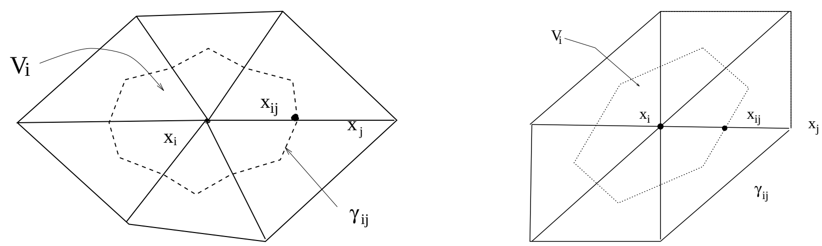

FIG. 1. Control volumes with medicenters as internal point and interface γ_{ij} of V_i and V_j.

three edges $\overline{x_i x_j}$ of K a point x_{ij} is selected; then q is connected with the points x_{ij} by straight lines γ_{ij}.

Thus, around each vertex $x_j \in N_h^0$, we associate the control volume $V_j \in T_h^*$, which consists of the union of the sub-elements $K \in T_h$, which have x_j as a vertex. Also let γ_{ij} denote the interface of two control volumes V_i and V_j: $\gamma_{ij} = V_j \cap V_j$, $j \in \Pi(i)$ (see Figure 1 and 2).

We call the partition T_h^* *regular or quasiuniform* if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \leq \text{meas}(V_i) \leq Ch^2, \qquad \text{for all} \qquad V_i \in T_h^*.$$

Here h is the maximal diameter of all elements $K \in T_h$ In this paper we shall deal with regular triangulations T_h^*.

The partition T_h^* is said to be *symmetric* if $x_{ij} = \gamma_{ij} \cap \overline{x_i x_j}$ is the middle point of the line segment $\overline{x_i x_j}$, and x_{ij} is the middle point of γ_{ij} or γ_{ij} has two perpendicular axes of symmetry and x_{ij} is their intersection point.

There are various ways of introducing regular control volume grids T_h^*. Widely used in the finite volume element method are the following two partitions, which we shall employ in our paper (see Figures 1 and 2).

In the first (and most popular) control volume partition the point q is chosen to be the medicenter (the center of gravity or centroid) of the finite element K and the points x_{ij} are chosen to be the midpoints of the edges of K (see Figure 1). This type of control volumes can be introduced for any finite element partition T_h and lead to relatively simple calculations for both 2- and 3-D problems. Besides, if the finite element partition T_h is locally regular, i.e. there is a constant C such that $Ch_K^2 \leq \text{meas}(K) \leq h_K^2$, $\text{diam}(K) = h_K$ for all elements $K \in T_h$ then the finite volume partition T_h^* is also locally regular.

In this paper we shall use also the construction of the control volumes in which the point q is the circumcenter of the element K, i.e. the center of the circumscribed circle of K and x_{ij} are the midpoints of the edges of K. This type of control volumes form the so-called Voronoi meshes. Then obviously, γ_{ij} are the perpendicular bisectors of the three edges of K (see Figure 2). This construction requires that all finite elements are triangles of acute type, which we shall assume whenever such triangulation is used.

We are now ready to define the finite element space S_h of linear elements:

$$S_h = \{v \in C(\Omega): \ v|_K \ \text{ is linear for all } \ K \in T_h \ \text{ and } v|_{\partial\Omega} = 0\}$$

and its dual volume element space S_h^*:

$$S_h^* = \{v \in L^2(\Omega): \ v|_V \ \text{ is constant for all } \ V \in T_h^* \ \text{ and } v|_{\partial\Omega} = 0\}.$$

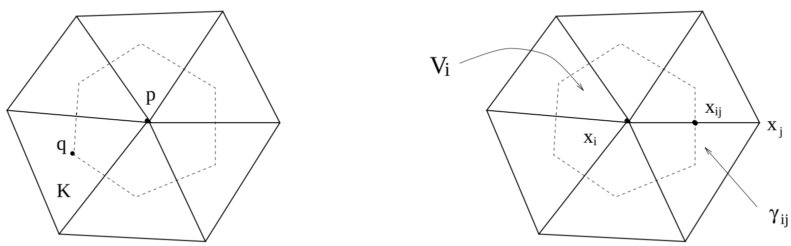

FIG. 2. Control volumes with circumcenters as internal points (Voronoi meshes) and interface γ_{ij} of V_i and V_j.

Obviously, $S_h = \text{span}\{\phi_i(x) : x_i \in N_h^0\}$ and $S_h^* = \text{span}\{\chi_i(x) : x_i \in N_h^0\}$, where ϕ_i are the standard nodal linear basis function associated with the node x_i and χ_i are the characteristic functions of the volume V_i. Let $I_h : C(\Omega) \to S_h$ be the interpolation operator and $I_h^* : C(\Omega) \to S_h^*$ be the piece-wise constant interpolation operators, respectively. That is

$$I_h u = \sum_{x_i \in N_h} u(x_i)\phi_i(x), \qquad \text{and} \qquad I_h^* u = \sum_{x_i \in N_h} u(x_i)\chi_i(x).$$

The semi-discrete finite volume element approximation u_h of (2.0) is a solution to the problem: Find $u_h(t) \in S_h$ for $t > 0$ such that

$$\begin{aligned} (u_{h,t}, v_h) + A(u_h, v_h) + \int_0^t B(t, s; u_h(s), v_h) ds &= (f, v_h), \qquad v_h \in S_h^* \\ u_h(0) &= u_{0,h} \in S_h, \end{aligned} \tag{2.2}$$

or

$$(u_{h,t}, I_h^* v_h) + A(u_h, I_h^* v_h) + \int_0^t B(t, s; u_h(s), I_h^* v_h) ds = (f, I_h^* v_h), \qquad v_h \in S_h. \tag{2.3}$$

Here the bilinear forms $A(u, v)$ is defined

$$A(u,v) = \begin{cases} -\displaystyle\sum_{x_i \in N_h} v_i \int_{\partial V_i} A\nabla u \cdot \mathbf{n} dS_x, & (u,v) \in H_0^1 \cap H^2 \times S_h^*, \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v dx, & (u,v) \in H_0^1 \times H_0^1, \end{cases} \tag{2.4}$$

where $\mathbf{n}$ denotes the outer-normal direction to the domain under consideration. The form $B(\cdot, \cdot)$ is defined in a similar way.

Remark 2.3. *We use the same notation for the bilinear forms A and B defined in two different ways on the pair of spaces $H_0^1 \times H_0^1$ and $H_0^1 \cap H^2 \times S_h^*$, correspondingly. We hope that this will not lead to serious confusion while it simplifies tremendously the notations and the overall exposition of the material.*

There is one more reason to use this notation. Namely, we have the following important result, due to [12, 16]:

Lemma 2.1. *If the matrix A is constant over each element $K \in T_h$ then for all $u_h, v_h \in S_h$ the following equality holds true:*

$$A(u_h, v_h) = A(u_h, I_h^* v_h).$$

Proof: Take $K \in T_h$ and let V_i, $i = 1, 2, 3$ be the the three volumes intersecting K. Then for $u_h, v_h \in S_h$ obviously we have

$$\begin{aligned} 0 &= \int_K \nabla(A\nabla u_h)\ I_h^* v_h dx = \sum_i \int_{\partial(V_i \cap K)} A\nabla u_h \cdot \mathbf{n}\ I_h^* v_h dx \\ &= \int_{\partial K} A\nabla u_h \cdot \mathbf{n}\ I_h^* v_h dx - \sum_i v_i \int_{\partial V_i \cap K} A\nabla u_h \cdot \mathbf{n} dx \\ &= \int_{\partial K} A\nabla u_h \cdot \mathbf{n}\ v_h dx - \sum_i v_i \int_{\partial V_i \cap K} A\nabla u_h \cdot \mathbf{n} dx. \end{aligned}$$

On the other hand,

$$0 = \int_K \nabla(A\nabla u_h)\ v_h dx = \int_{\partial K} A\nabla u_h \cdot \mathbf{n}\ v_h dx - \int_K A\nabla u_h \cdot \nabla v_h dx.$$

Summing over all elements of the partition $\mathcal{T}_h$ we get the desired result.

Next, we define the fully discrete time stepping schemes. Let $\Delta t > 0$ be a time-step size and $t_n = n\Delta t$, $n = 0, 2, \cdots$, $g^n = g(t_n)$, and $\partial g^n = (g^n - g^{n-1})/\Delta t$.

The backward Euler scheme is defined to be the solution of $u_h^n \in S_h$ such that

$$\begin{aligned} (\frac{u_n^n - u_h^{n-1}}{\Delta t_n}, I_h^* v_h) + A(u_h^n, I_h^* v_h) + \sum_{k=0}^{n-1} \omega_{n,k} B(t_n, t_k; u_h^k, I_h^* v_h) &= (f^n, I_h^* v_h), \quad (2.5) \\ u_h^0(0) &= u_{0,h} \in S_h, \end{aligned}$$

where $\omega_{n,j}$ are the weights and the quadrature error is given by for any smooth functions g and M and its error

$$q_h(g) = \int_0^{t_n} M(t_n, s) g(s) ds - \sum_{j=1}^{n-1} \omega_{n,j} M_{n,j} g(t_j)$$

satisfies

$$|q_n(g)| \le C\Delta t \int_0^{t_n} (|g| + |g'|) dt.$$

C. Some Auxiliary Results

Here we recall several key properties of the bilinear forms defined in B..We first define some discrete norms on S_h and S_h^*:

$$\begin{aligned} |u_h|_{0,h}^2 &= (u_h, u_h)_{0,h}, \text{ with } (u_h, v_h)_{0,h} = \sum_{x_i \in N_h} \text{meas}(V_i) u_i v_i = (I_h^* u_h, I_h^* v_h), \\ |u_h|_{1,h} &= \sum_{x_i \in N_h} \sum_{x_j \in \Pi(i)} \text{meas}(V_i) \left((u_i - u_j)/d_{ij}\right)^2, \\ ||u_h||_{1,h}^2 &= |u_h|_{0,h}^2 + |u_h|_{1,h}^2, \qquad |||u_h|||_0^2 = (u_h, I_h^* u_h), \end{aligned}$$

where $d_{ij} = d(x_i, x_j)$, the distance between x_i and x_j.

In the lemmas below we assume that the matrix $A(x)$ may have jumps, which are aligned with the finite element partition T_h and over each element the entries of the matrix $A(x)$ are C^1-functions. We also assume that T_h is e regular partition of Ω.

Lemma 2.2. *(see, e.g. [3, 16]) There exist two positive constants $C_0, C_1 > 0$, independent of h, such that*

$$\begin{aligned} C_0|v_h|_{0,h} &\leq ||v_h||_0 \leq C_1|v_h|_{0,h}, \qquad v_h \in S_h, \\ C_0|||v_h|||_0 &\leq ||v_h||_0 \leq C_1|||v_h|||_0, \qquad v_h \in S_h, \\ C_0||v_h||_{1,h} &\leq ||v_h||_1 \leq C_1||v_h||_{1,h}, \qquad v_h \in S_h. \end{aligned}$$

Lemma 2.3. *(see, e.g. [3, 16]) There exist two positive constants $C_0, C_1 > 0$, independent of h, and $h_0 > 0$ such that for all $0 < h \leq h_0$*

$$|A(u_h, I_h^* v_h)| \leq C_1||u_h||_{1,h}||v_h||_{1,h}, \qquad u_h, \ \ v_h \in S_h, \tag{2.6}$$
$$A(u_h, I_h^* u_h) \geq C_0||u_h||_{1,h}^2, \qquad u_h, \ \ v_h \in S_h. \tag{2.7}$$

Now we introduce linear functionals $l_{ij}(u)$, which will be used in the error analysis of the finite volume method:

$$l_{ij}(u) \ = \ -\int_{\gamma_{ij}} A\nabla(I_h u - u)\cdot \mathbf{n} dS_x, \tag{2.8}$$

where $\gamma_{ij} = V_i \cap V_j$.

The following estimate is a simple consequence of the Bramble-Hilbert lemma:

Lemma 2.4. *(see, e.g. [3, 16]) If $u \in H^2(\Omega)$ then there is a positive constant $C > 0$ independent of h, i and j such that for $e_{ij} = \cup\{K \mid K \cap \gamma_{ij} \neq \emptyset, K \in T_h\}$,*

$$|l_{ij}(u)| \leq Ch||A||_{0,\infty}|u|_{2,e_{ij}}. \tag{2.9}$$

Lemma 2.5. *Assume that T_h is a regular, then there exists a positive constant $C > 0$, independent of h, such that*

$$|A(u - I_h u, I_h^* v_h)| \leq Ch||u||_2|v_h|_{1,h}, \qquad v_h \in S_h. \tag{2.10}$$

Proof: ¿From [3, 12, 16] we see that

$$\begin{aligned} A(u - I_h u, I_h^* v_h) &= \sum_{x_i \in N_h}\sum_{j\in\Pi(i)} v_i \int_{\partial V_i} (-A\nabla(u - I_h u))\cdot \mathbf{n} dS_x \\ &= \frac{1}{2}\sum_{x_i \in N_h}\sum_{j\in\Pi(i)} (l_{ij}(u)v_i + l_{ji}(u)v_j) = \frac{1}{2}\sum_{x_i \in N_h}\sum_{j\in\Pi(i)} l_{ij}(u)(v_i - v_j) \end{aligned}$$

so that it follows from Cauchy inequality and Lemma 2.3 that

$$|A(u - I_h u, I_h^* v_h)| \ \leq \ C\left(\sum_{x_i \in N_h}\sum_{j\in\Pi(i)} l_{ij}^2\right)^{1/2}\left(\sum_{x_i \in N_h}\sum_{j\in\Pi(i)} (v_i - v_j)^2\right)^{1/2}$$

$$\leq \; Ch||A||_{0,\infty}||u||_2||v_h||_{1,h}.$$

In case of symmetric partitions and smooth solutions, due to the cancelation in the local truncation error, one can get higher order approximations by the same finite elements. Namely, we can prove:

Lemma 2.6. *Assume that T_h is a regular and symmetric, then there exists a positive constant $C > 0$, independent of h, such that*

$$|A(u - I_h u, I_h^* v_h)| \leq Ch^2||u||_3|v_h|_{1,h}, \qquad v_h \in S_h. \tag{2.11}$$

Proof: By [16], we have in this case

$$|l_{ij}(u)| \leq Ch^{\mu-1/2}||u||_{\mu,e_{ij}}, \qquad 3/2 \leq \mu \leq 3.$$

Thus, Lemma 2.6 follows as before.

In fact, if the triangulation T_h is regular and any two adjacent elements forms an approximate parallelogram, then it can be proved that [13], [16]:

$$|A(u - I_h u, I_h^* v_h)| \leq Ch^2\left(||u||_3 + ||u||_{2,\infty}\right)|v_h|_{1,h}, \qquad v_h \in S_h.$$

This means that almost symmetric grids have the same convergence rates (for smooth solutions) as the symmetric ones.

For any fixed $0 < t \leq J$ one can define the Ritz projection $R_h u$ of function $u(x,t)$ where the operator $R_h : H_0^1 \cap H^2 \to S_h$ so that

$$A(u - R_h u, I_h^* v_h) = 0, \qquad \text{for all} \qquad v_h \in S_h. \tag{2.12}$$

Remark 2.4. *The results of the above lemmas can be summarized as follows:*
(a) if the partition T_h is regular (quasiuniform) and u is H^2-regular, then

$$||u - R_h u||_1 \leq C||u||_2;$$

(b) if the partition is regular and symmetric and u is H^3-regular, then

$$||u - R_h u||_{1,h} \leq Ch^2||u||_3.$$

However, these estimates for the Ritz-projection will lead to suboptimal error estimates for the finite volume element solution of the integro-differential equation. In order to obtain optimal estimates we need a projection which takes into account also the integral term. This type of projection has been called by Cannon and Lin [4] Ritz-Volterra projection and was used in the context of the finite element method.

D. Ritz-Volterra Projection and Its Properties

In this subsection we define the Ritz-Volterra projection, of a function defined on the cylinder $\Omega \times J$ and study its approximation properties in various norms.

The Ritz-Volterra projection $V_h : L^\infty(H_0^1 \cap H^2) \to L^\infty(S_h)$ is defined for $0 \leq t \leq T$ by

$$A(u - V_h u, I_h^* v_h) + \int_0^t B(t,s; u(s) - V_h u(s), I_h^* v_h)ds = 0, \qquad v_h \in S_h. \tag{2.13}$$

Theorem 2.1. *Assume that the mesh T_h^* is regular and $D_t^l u \in L^\infty(H^2)$ for all $0 \leq l \leq k$ for some integer $k \geq 0$. Then, the Ritz-Volterra projection $V_h u$ is well-defined and satisfies for all $t \geq 0$ and $k \geq 0$,*

$$||D_t^l(u - V_h u)||_1 \leq Ch \left(\sum_{j=0}^{l} ||D_t^j u||_2 + \int_0^t \sum_{j=0}^{l} ||D_t^j u(s)||_2 ds \right). \tag{2.14}$$

In addition, if T_h^* is also symmetric and $D_t^l u \in L^\infty(H^3)$, then we have

$$||D_t^l(I_h u - V_h u)||_1 \leq Ch^2 \left(\sum_{j=0}^{l} ||D_t^j u||_3 + \int_0^t \sum_{j=0}^{l} ||D_t^j u(s)||_3 ds \right). \tag{2.15}$$

Proof: ¿From Lemma 2.3 and 2.5 we see easily that for $w = I_h u - V_h u \in S_h$,

$$\begin{aligned} c_0 ||I_h u - V_h u||_1^2 &\leq A(I_h u - V_h u, I_h^* w) = A(u - V_h u, I_h^* w) + a(I_h u - u, I_h^* w) \\ &= -\int_0^t B(t, s; u(s) - V_h u(s), I_h w) ds + A(I_h u - u, I_h^* w) \\ &= -\int_0^t B(t, s; u(s) - I_h u(s), I_h^* w) ds \\ &\quad -\int_0^t B(t, s; I_h u(s) - V_h u(s), I_h^* w) ds + A(I_h u - u, I_h^* w) \\ &\leq Ch \left(||u||_2 + \int_0^t ||u|| ds_2 \right) ||w||_1 + C \int_0^t ||I_h u - V_h u||_1 ds \; ||w||_1. \end{aligned}$$

Thus,

$$||I_h u - V_h u||_1 \leq Ch \left(||u||_2 + \int_0^t ||u||_2 ds_2 \right) + C \int_0^t ||I_h u - V_h u||_1 ds$$

and then the case $l = 0$ of (2.14) follows from Gronwall's Lemma. The case $l \geq 1$ can be proved in a similar way [5, 10]. As far as the superconvergence estimate (2.15) is concerned the proof is the same as before, instead of (2.10) we use (2.11).

Now we consider an estimate in L^2-norm for the Ritz-Volterra projection which is optimal respect to the order of convergence and requires $W^{3,p}$-regularity of the solution. Therefore, this estimate is suboptimal respect to the regularity of the solution and makes sense for p close to 1. Namely we prove the following result:

Theorem 2.2. *Assume that the partition T_h is a regular and for some $p > 1$ and an integer $k \geq 0$, $D_t^l u \in L^\infty(W^{3,p}(\Omega))$ for $0 \leq l \leq k$. Then there exists a positive constant $C > 0$ independent of h such that*

$$||D_t^l(u - V_h u)|| \leq Ch^2 \sum_{j=0}^{l} \left(||D_t^j u||_{3,p} + \int_0^t ||D_t^j u||_{3,p} ds \right) \quad for \quad 0 \leq l \leq k. \tag{2.16}$$

Proof: Note, that somewhat higher order regularity is required than the standard optimal error estimates in the finite element method. Namely, we have assumed that u is in $W^{3,p}$ for $p > 1$ which is more than H^2 but less than H^3.

We shall prove first (2.16) for $k=0$, i.e. the estimate

$$||u - V_h u|| \le Ch^2 \left(||u||_{3,p} + \int_0^t ||u||_{3,p} ds \right).$$

Let $w \in H_0^1 \cap H^2$ be the solution of the operator equation $\mathcal{A}w = u - V_h u$, i.e.

$$A(w,v) = (u - V_h u, v) \quad \text{for all} \quad v \in H_0^1(\Omega).$$

Then choosing $v = u - V_h u$ we get

$$\begin{aligned}
& ||u - V_h u||^2 = A(u - V_h u, w) = A(u - V_h u, w - I_h w) \\
& + A(u - V_h u, I_h w - I_h^* w) + \int_0^t B(t,s; u(s) - V_h u(s), I_h w - I_h^* w) ds \qquad (2.17) \\
& + \int_0^t B(t,s; u(s) - V_h u(s), I_h w - w) ds + \int_0^t B(t,s; u(s) - V_h u(s), w) ds \\
= \ & J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

We now estimate J_i terms. For the first term J_1, we have from Theorem 2.6 for $l = 0$ and regularity for elliptic problem $\mathcal{A}w = u - V_h u$ (that is $||w||_2 \le C||u - V_h u||$) that

$$\begin{aligned}
|J_1| \ &= \ |A(u - V_h u, w - I_h w)| \le Ch^2 \left(||u||_2 + \int_0^t ||u||_2 ds \right) ||w||_2 \\
&\le \ Ch^2 \left(||u||_2 + \int_0^t ||u||_2 ds \right) ||u - V_h u||.
\end{aligned}$$

Similarly, for the last two terms J_4 and J_5 we have

$$\begin{aligned}
|J_4| \ &\le \ |\int_0^t B(t,s; u(s) - V_h u(s), I_h w - w) ds| \le Ch^2 \int_0^t ||u||_2 ds ||u - V_h u||, \\
|J_5| \ &= \ |\int_0^t B(t,s; u(s) - V_h u(s), w) ds| \\
&= \ \left| \int_0^t (u - V_h, B^*(t,s) w) ds \right| \le C \int_0^t ||u - V_h u|| ds ||u - V_h u||,
\end{aligned}$$

where $B^*(t,s)$ is the adjoint operator of $B(t,s)$.

We now consider the remaining two terms J_2 and J_3. We again introduce $\bar{A}(x)$ as the average value of $A(x)$ over each finite element. Then

$$\begin{aligned}
& A(u - V_h u, I_h w) = \sum_K \int_K A \nabla (u - V_h u) \cdot \nabla I_h w dx \\
= \ & \sum_K \int_K (A - \bar{A}) \nabla (u - V_h u) \cdot \nabla I_h w dx + \sum_K \int_K \bar{A} \nabla (u - V_h u) \cdot \nabla I_h w dx \\
= \ & \sum_K \int_K (A - \bar{A}) \nabla (u - V_h u) \cdot \nabla I_h w dx - \sum_K \int_{\partial K} \bar{A} \nabla (u - V_h u) \cdot \mathbf{n} I_h w dS \\
& - \sum_K \int_K (\nabla \cdot \bar{A} \nabla (u - V_h u)) \, I_h w dx,
\end{aligned}$$

and similarly that

$$
\begin{aligned}
A(u - V_h u, I_h^* w) &= -\sum_{x_i \in N_h} \int_{\partial V_i} A\nabla(u - V_h u) \cdot \mathbf{n} dS \; w_i \\
&= -\sum_K \sum_{x_i \in N_h} \int_{\partial V_i \cap K} A\nabla(u - V_h u) \cdot \mathbf{n} I_h^* \; w dS \\
&= -\sum_K \sum_{x_i \in N_h} \int_{\partial V_i \cap K} (A - \bar{A})\nabla(u - V_h u) \cdot \mathbf{n} \; I_h^* w dS \\
&\quad -\sum_K \sum_{x_i \in N_h} \int_{\partial V_i \cap K} \bar{A}\nabla(u - V_h u) \cdot \mathbf{n} \; I_h^* w dS \\
&= -\sum_K \sum_{x_i \in N_h} \int_{\partial V_i \cap K} (A - \bar{A})\nabla(u - V_h u) \cdot \mathbf{n} \; I_h^* w dS \\
&\quad -\sum_K \int_K (\nabla \cdot \bar{A}\nabla(u - V_h u)) \; I_h^* w dx - \sum_K \int_{\partial K} \bar{A}\nabla(u - V_h u) \cdot \mathbf{n} \; I_h^* w dS
\end{aligned}
$$

Thus J_3 can be expressed by

$$
\begin{aligned}
J_3 &= \sum_K \int_K (A - \bar{A})\nabla(u - V_h u) \cdot \nabla I_h w dx \\
&\quad -\sum_K \sum_{x_i \in N_h} \int_{\partial V_i \cap K} (A - \bar{A})\nabla(u - V_h u) \cdot \mathbf{n} \; I_h^* w dS \\
&\quad -\sum_K \int_K (\nabla \cdot \bar{A}\nabla(u - V_h u))(I_h w - I_h^* w) dx \\
&\quad -\sum_K \int_{\partial K} \bar{A}\nabla(u - V_h u) \cdot \mathbf{n} \; (I_h w - I_h^* w) dS \\
&= J_{31} + J_{32} + J_{33} + J_{34}.
\end{aligned}
$$

For J_{31}, we see from Theorem 2.6 that

$$
\begin{aligned}
|J_{31}| &\le Ch||A||_{1,\infty} \sum_K \int_K |\nabla(u - V_h u)| \, |\nabla I_h w| dx \\
&\le Ch^2 \left(||u||_2 + \int_0^t ||u||_2 ds \right) ||u - V_h u||,
\end{aligned}
$$

and for J_{32} from inverse assumptions and Theorem 2.6 that

$$
\begin{aligned}
|J_{32}| &= \left| \frac{1}{2} \sum_{x_i \in N_h} \sum_{j \in \Pi(i)} (w_i - w_j) \int_{\gamma_{ij}} (A - \bar{A})\nabla(u - I_h u + I_h u - V_h u) \cdot \mathbf{n} dS \right| \\
&\le Ch^2 \sum_K \left(||u||_{2,K} + \int_0^t ||u||_{2,K} ds \right) |I_h w|_{1,K} \\
&\le Ch^2 \left(||u||_2 + \int_0^t ||u||_2 ds \right) ||u - V_h u||.
\end{aligned}
$$

For J_{33}, we notice that

$$\int_K (I_h w - I_h^* w) dx = 0 \quad \text{for} \quad K \in T_h,$$

so we can construct $\bar{u}$ some quadratic interpolant (or approximant) of u on K, so that

$$\begin{aligned} |J_{33}| &= \left| \sum_K \int_K (\nabla \cdot \bar{A} \cdot \nabla(u - \bar{u}))(I_h w - I_h^* w) dx \right| \\ &\le Ch \sum_K ||u||_{3,p,K}|| \, ||I_h w - I_h^* w||_{0,q} \le Ch^2 \sum_K ||u||_{3,p,K}|| \, ||w||_{1,q,K} \\ &\le Ch^2 \left(||u||_{3,p} + \int_0^t ||u||_{3,p} ds \right) ||u - V_h u||. \end{aligned}$$

Here we have used the the fact that we are interested in p close to 1 and therefore $q > 2$ and the interpolants are well defined and sable in $W^{1,q}$.

For J_{34}, we have that

$$\begin{aligned} J_{34} &= -\sum_K \frac{1}{3} \sum_{K'} \int_{\partial K \cap \partial K'} (\bar{A_K} - \bar{A_{K'}}) \nabla(u - I_h u) \cdot \mathbf{n} \, (I_h w - I_h^* w_h) dS \\ &= Ch||A||_{1,\infty} \sum_K \int_{\partial K} |\nabla(u - I_h u)| \, |I_h w - I_h^* w| dS \le Ch^2 ||u||_2 ||u - V_h u||. \end{aligned}$$

After employing the embedding $||u||_2 \le C||u||_{3,p}$ we obtain from the above estimates that

$$|J_3| \le Ch^2 \left(||u||_{3,p} + \int_0^t ||u||_{3,p} ds \right) ||u - V_h u||.$$

Similarly, it follows for J_4 that

$$|J_4| \le Ch^2 ||u - V_h u|| \int_0^t ||u||_{3,p} ds.$$

Now substituting all these estimates for J_i into (2.16), we get

$$||u - V_h u|| \le Ch^2 \left(||u||_{3,p} + \int_0^t ||u||_{3,p} ds \right) + C \int_0^t ||u - V_h u|| ds,$$

which together with Gronwall's inequality yields

$$||u - V_h u|| \le Ch^2 \left(||u||_{3,p} + \int_0^t ||u||_{3,p} ds \right).$$

The proof for $k \ge 1$ is similar to the analysis above, so we omit it.

E. The Regularized Green Function and Its Estimates

We have shown optimal H^1 norm and superconvergence of the gradients for the Ritz-Volterra projection associated with finite volume element approximations, as our main goal is to study the finite volume element methods for parabolic integro-differential equations some kind of L^2 error estimates are desirable if not optimal both with respect to the order of convergence and regularity of the solution.

We consider L^2 error estimates and maximum norm error estimates for Ritz-Volterra projection in next subsections. Our techniques are the regularized Green functions estimates and connection of the solutions of finite volume element and finite element approximations.

Following [15] we introduce the regularized Green function to the integro-differential equation, which is used in maximum norm estimates and superconvergence for finite element approximations parabolic integro-differential equations.

We define a function $G^z(t) \equiv G(x,t;z) \in L^2(H_0^1(\Omega), J;) \cap H^2(\Omega))$, where $x = (x_1, x_2)$ is the space variable and $z = (z_1, z_2)$ is a point in Ω, to be the solution of the equation

$$\mathcal{A}G^z(t) + \int_t^T \mathcal{B}^*(\tau,t)G^z(\tau)d\tau = \delta_h^z(x)\phi(t) \qquad \text{in } \ \Omega \times J, \tag{2.18}$$

where $\phi(t) \in C^\infty(0,T)$ and $\delta_h^z(x) \in S_h$ is a smoothed δ-function associated with the point z and the operator $\mathcal{B}^*$ is defined by $(\mathcal{B}^*u, v) = \int_\Omega B^T(t,\tau)\nabla u \cdot \nabla v dx = B^*(t,\tau : u, v)$. They are required to satisfy the following properties:

$$\begin{aligned}&(\delta_h^z, v_h) = v_h(z), \qquad v_h \in S_h, \qquad \|\phi\|_{L^1(0,T)} \le 1, \qquad \text{and}\\ &|\delta_h^z(x)| \le Ch^{-2}, \quad \operatorname{supp}(\delta_h^z) \subset \{x;\ |x-z| \le Ch\}.\end{aligned}$$

The solution of the auxiliary problem (2.18) plays the same role as the regularized Green's function used in the L^∞-error analysis for the finite element method of the elliptic problem of second order [18, 27], though the function on the right hand side of (2.18) is not precisely a regularized δ-function in both space and time.

We note that the regularized Green function is $\phi(t)$ dependent, this in fact allows us certain flexibility to choose $\phi(t)$ arbitrarily according to our needs.

Let $G_h^z(t)$ be the finite element approximation of the regularized Green's function for $t \in J$, i.e.,

$$A(G^z(t) - G_h^z(t), \chi) + \int_t^T B^*(\tau, t; G^z(\tau) - G_h^z(\tau), \chi)d\tau = 0, \quad \chi \in S_h,$$

Following [27], for a given point $z \in \Omega$ we introduce $\partial_z G^z(t)$ and its finite element approximation $\partial_z G_h^z(t)$ defined for any $w \in H_0^1(\Omega)$:

$$A(\partial_z G^z(t) - \partial_z G_h^z(t), \chi) + \int_t^T B^*(\tau, t; \partial_z G^z(\tau) - \partial_z G_h^z(\tau), \chi)d\tau = 0, \quad \chi \in S_h.$$

The functions $G^z(t)$ and $\partial_z G^z(t)$ have the following property: for any $w \in H_0^1(\Omega)$

$$\begin{aligned} P_h w(z)\phi(t) &= A(G^z(t), w) + \int_t^T B^*(\tau, t; G^z(\tau), w)d\tau, &(2.19)\\ \partial_z P_h w(z)\phi(t) &= A(\partial_z G^z(t), w) + \int_t^T B^*(\tau, t; \partial_z G^z(\tau), w)d\tau, &(2.20)\end{aligned}$$

where P_h is L^2 projection operator.

The following estimates for Ritz projection for regular triangulations T_h have been proven in [15, 26]:

$$\|G^z(t) - G_h^z(t)\|_{1,1} \le Ch\log\frac{1}{h}\ \ (1 + |\phi(t)|), \tag{2.21}$$

$$\|\partial_z G^z(t) - \partial_z G_h^z(t)\|_{1,1} \le C(1+|\phi(t)|), \qquad t \in J, \tag{2.22}$$

$$||G^z(t)||_{1,q} \le C_q(1+|\phi(t)|), \qquad 1 \le q < 2, \qquad t \in J, \tag{2.23}$$

$$||\partial_z G^z(t)||_{1,1} \le C\log\frac{1}{h}(1+|\phi(t)|), \qquad t \in J. \tag{2.24}$$

with constant C independent of h, z, and $\phi(t)$.

Remark 2.5. *If $B(t,s)=0$ then the above results are quite standard for the regularized Green function estimates with $\phi(t)=1$. Estimates (2.21) and (2.22) are proved in [15] and (2.22), (2.23), and (2.24) as well as other estimates are shown in [26].*

Lemma 2.7. *Assume that $f(t),\ g(t) \in L^1(0,T_0)$ and there exists $C>0$ such that for any non-negative $\phi(t) \in C^\infty(0,T)$, $0<T\le T_0$,*

$$|\int_0^T f(t)\phi(t)dt| \le C\int_0^T g(t)(1+\phi(t))dt, \qquad 0<T<T_0,$$

then, we have

$$|f(t)| \le C\left\{g(t)+\int_0^t g(\tau)d\tau\right\}, \qquad \forall t\in(0,T_0), a.e.$$

Proof: Take $\mu>0$ and let

$$\phi_\mu(t,t_0) = \begin{cases} (C_\mu)^{-1}\exp\left(-\frac{\mu^2}{\mu^2-|t-t_0|^2}\right), & |t-t_0|<\mu, \\ 0, & |t-t_0|\ge\mu, \end{cases}$$

where t_0 is any fixed point in $(0,T)$ and $C_\mu = \mu\int_{|t|<1}\exp\left(-\frac{1}{1-t^2}\right)dt$. We see easily that for almost all $t_0 \in (0,T)$,

$$f(t_0) = \lim_{\mu\to 0}\int_0^T f(t)\phi_\mu(t,t_0)dt, \qquad f\in C^\infty(0,T).$$

Thus, if we take $f_n(t)\in C^\infty(0,T)$ such that $f_n(t)\to f(t)$ as $n\to\infty$ in $L^1(0,T)$, then the result is true for all $f_n(t)$. Therefore it is true for $f(t)$ via a limiting procedure.

F. L^∞-Error Estimates for Ritz-Volterra Projection

In this subsection we prove maximum norm error estimates for Ritz-Volterra Projection. We first obtain some auxiliary results and then in Theorem 2.7 we prove the main result.

For any matrix function $A(x)$, we introduce $\bar{A}(x)$ as a piecewise constant matrix defined by

$$\bar{A}(x) = \frac{1}{\text{meas}(K)}\int_K A(y)dy, \qquad x\in K\in T_h.$$

First observe that

$$A(u-V_hu, I_h^*v_h) + \int_0^t B(t,s;u(s)-V_hu(s), I_h^*v_h)ds$$

$$
\begin{aligned}
&= -\sum_{x_i\in N_h}\int_{\partial V_i}\left\{A\nabla(u-I_hu)\cdot\mathbf{n}+\int_0^t B(t,s)\nabla(u-I_hu)\cdot\mathbf{n}ds\right\}v_i\\
&- \sum_{x_i\in N_h}\int_{\partial V_i}\left\{(A-\bar{A})\nabla(I_hu-V_hu)\cdot\mathbf{n}+\int_0^t (B(t,s)-\bar{B}(t,s))\nabla(I_hu-V_hu)\cdot\mathbf{n}ds\right\}v_i\\
&- \sum_{x_i\in N_h}\int_{\partial V_i}\left\{\bar{A}\nabla(I_hu-V_hu)\cdot\mathbf{n}+\int_0^t \bar{B}(t,s)\nabla(I_hu-V_hu)\cdot\mathbf{n}ds\right\}v_i=0.
\end{aligned}
$$

By Lemma 2.1 the last term of the right hand side is equivalent to the standard bilinear form since $\bar{A}$ and $\bar{B}$ are piece-wise constant on each elements in T_h. Thus the above identity can be rewritten as

$$
\begin{aligned}
&A(I_hu-V_hu,v_h)+\int_0^t B(t,s;I_hu(s)-V_hu(s),v_h)ds\\
=\ &-\frac{1}{2}\sum_{x_i\in N_h}\sum_{j\in\Pi(i)}\left(l_{ij}^A+\int_0^t l_{ij}^B ds\right)(v_i-v_j)\\
&-\frac{1}{2}\sum_{x_i\in N_h}\sum_{j\in\Pi(i)}\left(l_{ij}^{A-\bar{A}}+\int_0^t l_{ij}^{B-\bar{B}} ds\right)(v_i-v_j)\\
&+L^{A-\bar{A}}+\int_0^t L^{B-\bar{B}}ds,
\end{aligned}
\tag{2.25}
$$

where

$$
\begin{aligned}
l_{ij}^A &= -l_{ji}^A=\int_{\gamma_{ij}} A\nabla(u-I_hu)\cdot\mathbf{n}dS,\\
l_{ij}^{A-\bar{A}} &= -l_{ji}^{A-\bar{A}}=\int_{\gamma_{ij}} (A-\bar{A})\nabla(I_hu-V_hu)\cdot\mathbf{n}dS,\\
L^{A-\bar{A}} &= \sum_{K\in T_h}\int_K (A-\bar{A})\nabla(I_hu-V_hu)\cdot\nabla v_h dx,
\end{aligned}
$$

and l_{ij}^B, $l_{ij}^{B-\bar{B}}$ and $L^{B-\bar{B}}$ are defined similarly.

Assume that T_h is a regular and symmetric, then we have from Lemma 4.3 of [16] and Holder inequality that

$$
|l_{ij}^A|\le Ch^{s-1}||u||_{s,e_{ij}}\le Ch^{s-2/p}||u||_{s,p,e_{ij}},\qquad 2<p\le\infty,\qquad 1+1/p<s\le 3,
$$

so that

$$
\begin{aligned}
\left|\sum_{x_i\in N_h}\sum_{j\in\Pi(i)} l_{ij}^A(v_i-v_j)\right| &\le \left(\sum_{x_i\in N_h}\sum_{j\in\Pi(i)}|l_{ij}^A|^p\right)^{1/p}\left(\sum_{x_i\in N_h}\sum_{j\in\Pi(i)}|v_i-v_j|^q\right)^{1/q}\\
&\le Ch^{s-1}||u||_{s,p}||v_h||_{1,q},
\end{aligned}
$$

where p and q are two conjugate numbers. A similar estimate holds for the integral term related to l_{ij}^B.

We also find from Theorem 2.6 and inverse estimates that

$$
|l_{ij}^{A-\bar{A}}|\ \le\ Ch||A||_{1,\infty}\mathrm{meas}(\gamma_{ij})||\nabla(I_hu-V_hu)||_\infty
$$

$$\leq \; Ch||\nabla(I_h u - V_h u)||_1 \leq Ch^2 \left(||u||_3 + \int_0^t ||u||_3 ds \right),$$

which implies that

$$\left| \sum_{x_i \in N_h} \sum_{j \in \Pi(i)} l_{ij}^{A-\bar{A}}(v_i - v_j) \right| \leq Ch^2 \left(||u||_3 + \int_0^t ||u||_3 ds \right) ||v_h||_{1,1}$$

Similarly,

$$\left| L^{A-\bar{A}} + \int_0^t L^{B-\bar{B}} ds \right| \leq Ch^2 \left(||u||_3 + \int_0^t ||u||_3 ds \right) ||v_h||_{1,1}.$$

Combining all estimates above we have that

$$\begin{aligned} &\left| \int_0^T \left\{ A(I_h u - V_h u, v_h) + \int_0^t B(t,s; I_h u(s) - V_h u(s), v_h) ds \right\} dt \right| \\ \leq \; & Ch^{s-1} \int_0^T \left(||u||_{s,p} + \int_0^s ||u(\tau)||_{s,p} d\tau \right) ||v_h||_{1,q} ds \\ & + Ch^2 \int_0^T \left(||u||_3 + \int_0^s ||u(\tau)||_3 d\tau \right) ||v_h||_{1,1} ds. \end{aligned} \tag{2.26}$$

Theorem 2.3. *Assume that T_h is regular and symmetric, $D_t^l u \in L^\infty(H_0^1 \cap H^3 \cap W^{\mu,p})$ for all $0 \leq l \leq k$ for some integer $k \geq 0$, some real $\mu > 1$ and $p > 2$. Then there exists positive $C > 0$, independent of h, such that for all $0 \leq l \leq k$*

$$||D_t^l(u - V_h u)||_{0,\infty} \; \leq \; Ch^{\mu-1} \sum_{j=0}^{l} \left(||D_t^j u||_{\mu,p} + \int_0^t ||D_t^j u(s)||_{\mu,p} ds \right), \tag{2.27}$$

$$\begin{aligned} ||D_t^l(I_h u - V_h u)||_{1,\infty} \; \leq \; & Ch^{\mu-1} \log \frac{1}{h} \sum_{j=0}^{l} \left(||D_t^j u||_{\mu,\infty} + \int_0^t ||D_t^j u(s)||_{\mu,\infty} ds \right) \\ & + Ch^2 \log \frac{1}{h} \sum_{j=0}^{l} \left(||D_t^j u||_3 + \int_0^t ||D_t^j u(s)||_3 ds \right). \end{aligned} \tag{2.28}$$

Proof: We first prove the estimate (2.27) for $l = 0$. Let $v_h = G_h^z$ into (2.25), we obtain using (2.19), integration by part, and interchanges of order of integration that

$$\begin{aligned} \left| \int_0^T (I_h u - V_h u)(z) \phi(s) ds \right| \; \leq \; & Ch^{\mu-1} \int_0^T \left(||u||_{\mu,p} + \int_0^s ||u(\tau)||_{\mu,p} d\tau \right) ||G_h^z||_{1,q} ds \\ & + Ch^2 \int_0^T \left(||u||_3 + \int_0^s ||u(\tau)||_3 d\tau \right) ||G_h^z||_{1,1} ds \\ \leq \; & Ch^{\mu-1} \int_0^T \left(||u||_{\mu,p} + \int_0^s ||u(\tau)||_{\mu,p} d\tau \right) (1 + \phi) \\ & + Ch^2 \int_0^T \left(||u||_3 + \int_0^s ||u(\tau)||_3 d\tau \right) (1 + \phi) ds. \end{aligned}$$

As in the definition of Green function, T can be chosen to be any fixed point big than t, we see from Lemma 2.7 that the above inequality can be replaced by

$$\begin{aligned}||I_h u - V_h u||_{0,\infty} \ \le\ & C_p h^{\mu-1} \int_0^t \left(||u||_{\mu,p} + \int_0^s ||u(\tau)||_{\mu,p} d\tau \right) \\ & + Ch^2 \int_0^t \left(||u||_3 + \int_0^s ||u(\tau)||_3 d\tau \right) ds,\end{aligned}$$

which is (2.27) for $l = 0$.

For superconvergence estimate (2.28) we set $v_h = \partial_z G_h^z$ in (2.25), and use Lemma 2.7 to obtain

$$\begin{aligned}||I_h u - V_h u||_{1,\infty} \ \le\ & Ch^{\mu-1} \log \frac{1}{h} \int_0^t \left(||u||_{\mu,\infty} + \int_0^s ||u(\tau)||_{\mu,\infty} d\tau \right) \\ & + Ch^2 \log \frac{1}{h} \int_0^t \left(||u||_3 + \int_0^s ||u(\tau)||_3 d\tau \right) ds\end{aligned}$$

so that (2.28) follows for $l = 0$.

The case of $l \ge 1$ can be shown in a similar way above. We now only prove (2.27) and (2.28) for $l = 1$ by assuming that $\mu = 3$ for simplicity. Let $\rho = u(t) - V_h u(t)$, we find by differentiating (2.13) that

$$A(\rho_t, I_h^* v_h) + B(t, t; \rho, I_h^* v_h) + \int_0^t B_t(t, s; \rho(s), I_h^* v_h) ds = 0, \qquad v_h \in S_h.$$

Now, we consider these terms individually. Observe that

$$\begin{aligned}A(\rho_t, I_h^* v_h) \ =\ & A(D_t(I_h u - V_h u), I_h^* v_h) - \frac{1}{2} \sum_{x_i \in N_h} \sum_{j \in \Pi(i)} l_{ij,t}^A (v_i - v_j) \\ & - \frac{1}{2} \sum_{x_i \in N_h} \sum_{j \in \Pi(i)} l_{ij,t}^{A-\bar{A}} (v_i - v_j) \\ =\ & A(D_t(I_h u - V_h u), v_h) + I_1 + I_2 + D_t L^{A-\bar{A}},\end{aligned}$$

where $l_{ij,t}^A$ and $l_{ij,t}^{A-\bar{A}}$ denote the time derivatives of l_{ij}^A, i.e.

$$\begin{aligned}l_{ij,t}^A &= l_{ij}(u - I_h u) = -\int_{\gamma_{ij}} A \nabla D_t (u - I_h u) \cdot \mathbf{n} dS, \\ l_{ij,t}^{A-\bar{A}} &= -\int_{\gamma_{ij}} (A - \bar{A}) \nabla D_t (I_h u - V_h u) \cdot \mathbf{n} dS, \\ D_t L^{A-\bar{A}} &= -\sum_{K \in T_h} \int_K (A - \bar{A}) \nabla D_t (I_h u - V_h) \cdot \nabla v_h dx.\end{aligned}$$

Clearly, we have from the same arguments above that

$$|I_1| \le Ch^2 ||u_t||_{3,\infty} ||v_h||_{1,1}$$

$$|I_2| \le Ch ||D_t(I_h u - V_h u)||_\infty ||v_h||_{1,1} \le Ch^2 \left(||u||_3 + ||u_t||_3 + \int_0^t ||u||_3 ds \right) ||v_h||_{1,1}$$

$$|D_t L^{A-\bar{A}}| \le Ch^2 \left(||u||_3 + ||u_t||_3 + \int_0^t ||u||_3 ds \right) ||v_h||_{1,1}.$$

Similarly, we write

$$\begin{aligned} B(t,t;\rho,I_h^* v_h) &= B(t,t;I_hu - V_hu),v_h) - \frac{1}{2}\sum_{x_i\in N_h}\sum_{j\in\Pi(i)} l_{ij}^{B(t,t)}(v_i - v_j) \\ &-\frac{1}{2}\sum_{x_i\in N_h}\sum_{j\in\Pi(i)} l_{ij}^{B(t,t)-B(\bar{t},t)}(v_i - v_j) + D_t L^{B(t,t)-B(\bar{t},t)}. \end{aligned}$$

The second, third and fourth terms of the above identity on the right hand side can be bounded in the same way as the bounds for I_1 and I_2, while the first term can be bounded as

$$\begin{aligned} |B(t,t;I_hu - V_hu),I_hv_h)| &\leq C||I_hu - V_hu||_{1,\infty}||v_h||_{1,q} \\ &\leq Ch^2||u||_{3,\infty}||v_h||_{1,1}, \end{aligned}$$

so that

$$|B(t,t;\rho,I_h^* v_h)| \leq Ch^2(||u||_{3,\infty} + \int_0^t ||u||_{3,\infty})||v_h||_{1,1}.$$

Similarly we have that

$$\left|\int_0^t B_t(t,s;\rho(s),I_h^* v_h)ds\right| \leq Ch^2\left(||u||_{3,\infty} + \int_0^t ||u||_{3,\infty}ds\right)||v_h||_{1,1}.$$

We thus have

$$A(D_t(I_hu - V_hu),v_h)| \leq Ch^2\left(||u||_{3,\infty} + \int_0^t ||u||_{3,\infty}ds\right)||v_h||_{1,1} \tag{2.29}$$

Recall from [27] that g^z is the regularized Green function for operator $\mathcal{A}$ and g_h^z is its finite element solution:

$$A(g^z - g_h^z, v_h) = 0 \qquad v_h \in S_h,$$

and

$$||g_h^z||_{1,1} \leq C \qquad and \qquad ||\partial g_h^z||_{1,1} \leq C\log\frac{1}{h}.$$

Set $v_h = g_h^z$ in (2.29), we have that

$$||D_t(I_hu - V_hu)||_{0,\infty} \;\leq\; Ch^2\left(||u_t||_{3,\infty} + \int_0^t ||u||_{3,\infty}ds\right),$$

where we have used

$$A(D_t(I_hu - V_hu),g_h^z) = D_t(I_hu - V_hu)(z).$$

Thus we have shown (2.27) for $l = 1$. For superconvergence of gradient, setting $v_h = \partial_z g_h^z$ [27] in (2.29) we obtain

$$||D_t(I_hu - V_hu)||_{1,\infty} \;\leq\; Ch^2\log\frac{1}{h}\left(||u_t||_{3,\infty} + ||u||_{3,\infty} + \int_0^t ||u||_3 ds\right).$$

We remark that there is no logarithm factor in maximum norm estimate for Ritz-Volterra projection associated with finite volume element approximation, which is different from the standard sharp maximum norm estimate for Ritz projection [18, 27] and Ritz-Volterra projection for finite element approximation, this may mainly be due to the higher order regularities required.

III. ERROR ANALYSIS FOR THE SEMI-DISCRETE PROBLEM

In this section we develop error estimates for the semi-discrete finite volume element method in various norms. The proofs are based on the results of Section II. for the error in Ritz-Volterra projection. Namely, in Subsection A. we derive L^2 and H^1 error estimates and in Subsection B. we derive maximum norm and superconvergence in maximum estimates.

A. Error Estimates in L^2- and H^1-norms

In this section we prove error estimates for the finite volume element approximation in L^2-, in H^1-norm, and an estimate for $||D_t(u-u_h)||$. The last one is needed in maximum-norm error estimates, derived in next sections.

Theorem 3.4. *Assume that T_h is regular and $u, D_t u \in L^\infty(H_0^1 \cap W^{3,p})$, for some $p > 1$ and for all $t \geq 0$. Assume also that the approximation $u_h(0)$ of the initial data satisfy $||u_h(0) - u_0|| \leq Ch^2||u_0||_2$. Then there exists a constant, independent of h and u such that for all $t \geq 0$*

$$||u - u_h|| \leq Ch^2 \left(||u_0||_{3,p} + \int_0^t ||u_t||_{3,p} ds \right). \tag{3.1}$$

Proof: Again we are interested in the case of p close to 1. Let $u - u_h = (u - V_h u) + (V_h u - u_h) = \rho + \theta$, where V_h is the Ritz-Volterra projection defined and analyzed in §2, so that we have

$$||\rho(t)|| \ \leq \ Ch^2 \left(||u||_{3,p} + \int_0^t ||u||_{3,p} ds \right), \tag{3.2}$$

$$||\rho_t(t)|| \ \leq \ Ch^2 \left(||u||_{3,p} + ||u_t||_{3,p} + \int_0^t ||u||_{3,p} ds \right), \tag{3.3}$$

and θ satisfies using (2.2) and (2.15) that

$$(\theta_t, I_h^* v_h) + A(\theta, I_h^* v_h) + \int_0^t B(t, s; \theta(s), I_h^* v_h) ds = -(\rho_t, I_h^* v_h), \qquad v_h \in S_h. \tag{3.4}$$

Set $v_h = \theta \in S_h$ to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |||\theta|||_0^2 + C_0 ||\theta||_1^2 \ &\leq \ C \int_0^t ||\theta||_1 ds \ ||\theta||_1 + ||\rho_t|| \ ||\theta|| \\ &\leq \ \frac{c_0}{2} ||\theta||^2 + C \int_0^t ||\theta||_1^2 ds + ||\rho_t|| \ ||\theta||, \end{aligned}$$

so that it follows from integration from 0 to t,

$$||\theta||^2 + \int_0^t ||\theta||_1^2 ds \le C \left(||\theta(0)||^2 + \int_0^t ||\rho_t||\, ||\theta|| ds + \int_0^t \int_0^t \int_0^\tau ||\theta(\tau)||_1^2 d\tau ds \right)$$

Thus, Gronwall's inequality leans to

$$||\theta||^2 \; + \; \int_0^t ||\theta||_1^2 ds \le C||\theta(0)||^2 + \frac{1}{2} \sup_{0<s<t} ||\theta(s)||^2 + C \left(\int_0^t ||\rho_t|| ds \right)^2,$$

and then, we have

$$||\theta|| \le C \left(||\theta(0)|| + C \int_0^t ||\rho_t|| ds \right) \le Ch^2 \left(||u_0||_{3,p} + \int_0^t ||u_t||_{3,p} ds \right).$$

Hence, Theorem 3.0 follows from (3.2)-(3.3) and the above inequality together with triangle inequality.

Now we derive the error estimate for the discrete H^1-norm, which can be interpreted as suprconvegence of the gradient of the solution at some particular points.

Theorem 3.5. *Assume that T_h is regular and $u, D_t u \in L^\infty(H_0^1 \cap H^3)$, for all $t \ge 0$. Assume also that the approximation $u_h(0)$ of the initial data satisfy $||u_h(0) - u_0|| \le Ch^2 ||u_0||_2$. Then there exists a constant, independent of h and u such that for all $t \ge 0$*

$$||V_h u - u_h||_{1,h} \le Ch^2 \left(||u_0||_3 + \int_0^t ||u_t||_3 ds \right). \tag{3.5}$$

Proof: We first note that $||u - u_h||_{1,h} \le ||I_h u - V_h u||_{1,h} + ||V_h u - u_h||_{1,h}$. Obviously $||I_h u - V_h u||_{1,h} = ||I_h u - V_h u||_1$ and this term has been already estimated in Theorem 2.6. Therefore, we need to show that

$$||V_h u - u_h||_{1,h} \le Ch^2 \left(||u_0||_3 + \int_0^t ||u_t||_3 ds \right).$$

Set $v_h = \theta_t$ in (3.4), it is easy to see that

$$\begin{aligned} |||\theta_t|||_0^1 + \frac{1}{2}\frac{d}{dt}||\theta||_1^2 \;&=\; (\rho_t, I_h^* \theta_t) - \int_0^t B(t,s;\theta(s), I_h^*\theta_t(t)) ds \\ &\le\; \frac{1}{2}||\rho_t||^2 + \frac{1}{2}|||\theta|||_1^2 + \frac{d}{dt}\int_0^t B(t,s;\theta(s), I_h^*\theta_t(t)) ds \qquad (3.6) \\ &\quad -B(t,t;\theta(t), I_h^*\theta(t)) - \frac{d}{dt}\int_0^t B_t(t,s;\theta(s), I_h^*\theta(t)) ds, \end{aligned}$$

where $B_t(t,s;\cdot,\cdot)$ is the bilinear form of time-differentiation in t of $B(t,s;\cdot,\cdot)$. We thus obtain from integration that

$$\begin{aligned} \int_0^t ||\theta_t||^2 ds + ||\theta||_1^2 \;&\le\; C\left(||\theta(0)||_1^2 + \int_0^t ||\rho_t||^2 ds \right) \\ &\quad + \int_0^t B(t,s;\theta(s), I_h^*\theta(t)) ds + C\int_0^t ||\theta||_1^2 ds \\ &\le\; C\left(||\theta(0)||_1^2 + \int_0^t ||\rho_t||^2 ds \right) + \frac{1}{2}||\theta||_1^2 + C\int_0^t ||\theta||_1^2 ds, \end{aligned}$$

By applying Gronwall's inequality, we have

$$\int_0^t ||\theta_t||^2 ds + ||\theta||_1^2 \le C\left(||\theta(0)||_1^2 + \int_0^t ||\rho_t||^2 ds\right)$$

Noticing that

$$||\theta_0||_1 \le ||V_h u_0 - u_0||_1 + ||u_0 - u_h(0)||_1 \le Ch^2 ||u_0||_3$$

it follows from Theorem 2.6 that

$$||\theta||_1^2 \le Ch^2 \left(||u_0||_3 + \int_0^t ||u_t||_3^2 ds\right).$$

In the following we shall estimate $||D_t(u - u_h)|| \equiv ||e_t||$, which is used to obtain maximum norms.

Theorem 3.6. *Under assumptions of Theorem 3.0 and the initial spatial approximation is chosen such that $u_h(0) = V_h u_0$. Then we have for $e(t) = u(t) - u_h(t)$,*

$$||e_t|| \le Ch^2 \left(||u_0||_3 + ||u_{tt}||_{3,p}\right) + \int_0^t \left(||u_t||_3 + ||u_{tt}||_{3,p}\right) ds. \tag{3.7}$$

Proof: By differentiating (3.4), it follows,

$$\begin{aligned} & (\theta_{tt}, I_h^* v_h) + A(\theta_t, I_h^* v_h) \\ = \;& -(\rho_{tt}, I_h^* v_h) - B_t(t,t;\theta, I_h^* v_h) - \int_0^t B_t(t,s;\theta(s), I_h^* v_h) ds, \end{aligned} \tag{3.8}$$

and then, by setting $v_h = \theta_t$, we have

$$\frac{1}{2}\frac{d}{dt}||\theta_t|| + C_0||\theta_t||_1^2 \le ||\rho_{tt}||\;||\theta_t|| + \frac{1}{2}C_0||\theta_t||_1^2 + C||\theta||_1^2 + \int_0^t ||\theta||_1^2 ds$$

Thus, it is to see from integration from 0 to t and the arguments similar to the proof of Theorem 3.0 that

$$||\theta_t||^2 \le C\left(||\theta_t(0)||^2 + |\int_0^t ||\rho_{tt}||^2 ds + \left(\int_0^t ||\theta||_1^2 ds\right)^{1/2}\right).$$

Since $\theta(0) = 0$ and using our assumption, we have from (3.4) that

$$(\theta_t(0), I_h^* v_h) = -(\rho_t(0), I_h^* v_h), \qquad v_h \in S_h$$

so that

$$||\theta_t(0)|| \le ||\rho_t(0)|| \le Ch^2 ||u_0||_3.$$

We now see $\theta_t(t)$ is bounded by the right hand side of (3.7). Therefore, (3.7) is proved by noticing $||e_t|| \le ||\rho_t|| + ||\theta_t||$.

B. L^∞- and $W^{1,\infty}$-Error Estimates

In this subsection the maximum norm error estimates and superconvergence of the gradient in maximum norm are derived here first time for finite volume element methods, that is, these results are even *new* for finite volume methods for elliptic equations. The key feature is that the relationship between finite element and finite volume element approximations derived in §2 for Ritz-Volterra projection and the regularized Green function.

Theorem 3.7. *Assume that the conditions of Theorem 3.0 are satisfied and $D_t^l u \in L^\infty(H_0^1 \cap H^3 \cap W^{\mu,p})$ for $p > 2$ and $1 \leq \mu \leq 3$. Then there is a constant $C > 0$, independent of h and u, such that*

$$||u - u_h||_{0,\infty} \leq Ch^{\mu-1}\left(||u||_{\mu,p} + \int_0^t ||u||_{\mu,p}ds\right) + Ch^2\left(||u||_3 + \int_0^t ||u||_3 ds\right) \quad (3.9)$$

Proof: Rewrite (3.4) as

$$A(\theta, I_h^* v_h) + \int_0^t B(t,s;\theta(s), I_h^* v_h)ds = (e_t, I_h^* v_h), \qquad v_h \in S_h.$$

or equivalently using Lemma 2.2 that

$$A(\theta, v_h) + \int_0^t B(t,s;\theta(s), v_h)ds = (e_t, I_h^* v_h) + L(\theta, v_h), \qquad v_h \in S_h. \qquad (3.10)$$

where

$$\begin{aligned} L(\theta, v_h) &= \frac{1}{2}\sum_{x_i \in N_h}\sum_{j \in \Pi}\left(l_{ij}^{A-\bar{A}}(\theta) + \int_0^t l_{ij}^{B-\bar{B}}(\theta)ds\right)(v_i - v_j) \\ &\quad + L^A(\theta, v_h) + L^B(\theta, v_h), \end{aligned}$$

and

$$\begin{aligned} l_{ij}^{A-\bar{A}}(\theta) &= \int_{\gamma_{ij}} (A - \bar{A})\nabla\theta \cdot \mathbf{n} dS \\ l_{ij}^{B-\bar{B}}(\theta) &= \int_{\gamma_{ij}} (B(t,s) - \bar{B}(t,s))\nabla\theta \cdot \mathbf{n} dS, \\ L^A(\theta, v_h) &= \sum_{K \in T_h}\int_K (A - \bar{A})\nabla\theta \cdot \nabla v_h dx \\ L^B(\theta, v_h) &= \int_0^t \left(\sum_{K \in T_h}\int_K (B - \bar{B})\nabla\theta \cdot \nabla v_h dx\right) ds \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \left|\frac{1}{2}\sum_{x_i \in N_h}\sum_{j \in \Pi} l_{ij}^{A-\bar{A}}(\theta)(v_i - v_j)\right| &\leq Ch^2||\nabla\theta||_\infty \sum_{x_i \in N_h}\sum_{j \in \Pi}|v_i - v_j| \\ &\leq Ch||\nabla\theta||_\infty ||v_h||_{1,1} \leq C||\nabla\theta||||v_h||_{1,1} \end{aligned}$$

and similarly,

$$\begin{aligned} \left|\frac{1}{2}\sum_{x_i \in N_h}\sum_{j \in \Pi}\int_0^t l_{ij}^{B-\bar{B}}(\theta)ds(v_i - v_j)\right| &\leq C\int_0^t ||\nabla\theta||ds||v_h||_{1,1} \\ |L^A(\theta, v_h) + L^B(\theta, v_h)| &\leq C\left(||\nabla\theta|| + \int_0^t ||\nabla\theta||ds\right)||v_h||_{1,1}. \end{aligned}$$

Thus, we obtain by substituting $v_h = G_h^z$ in (3.10) and using the above estimates together with integration form 0 to T and (2.20) that

$$\left|\int_0^T \theta(z,t)\phi(t)dt\right| \leq \int_0^T ||e_t||\,||G_h^z||ds + C\int_0^T \left(||\theta||_1 + \int_0^t ||\theta||_1 ds\right) ||G_h^z||_{1,1}dt$$
$$\leq \int_0^T ||e_t||(1+\phi(s))ds + C\int_0^T \left(||\theta||_1 + \int_0^t ||\theta||_1 ds\right)(1+\phi)dt.$$

So that Lemma 2.7 leads to

$$||\theta||_{0,\infty} \leq ||e_t|| + \int_0^t ||e_t||ds + C||\theta||_1 + C\int_0^t ||\theta||_1 ds.$$

Therefore, (3.9) follows from Theorem 2.7, the above inequality and the triangle inequality.

Theorem 3.8. *Under assumption of Theorem 3.2.1 and $u \in W^{3,\infty}$, we have for some constant $C > 0$, independent of h and u, such that*

$$||V_h u - u_h||_{1,\infty} \leq Ch^{\mu-1}\log\frac{1}{h}\left(||u||_{\mu,\infty} + \int_0^t ||u||_{\mu,\infty}ds\right) \qquad (3.11)$$
$$+Ch^2\log\frac{1}{h}\left(||u||_3 + \int_0^t ||u||_3 ds\right).$$

Proof: Setting $v_h = \partial_z G_h^z$ in (3.10), we see that (3.11) follows from Lemma 2.7 and the above analysis.

Remark 3.6. *Theorems 3.0 and 3.0, show that $||I_h u - u_h||_{1,\infty} = O(h^2 \log\frac{1}{h})$ provided that $u \in C^3$.*

Theorem 3.9. *Let u and u_h^n be the solutions of (2.0) and (2.2), respectively. Then we have some some constant $C > 0$, independent of h, Δt and u, such that*

$$||u(t) - u_h^n(t)|| \leq Ch^2\left(||u_0||_{3,p} + \int_0^t ||u_t||_{3,p}ds\right) \qquad (3.12)$$
$$+C\Delta t\int_0^{t_n} (||u_{tt}|| + ||u_t||_1)ds,$$

$$||u(t) - u_h^n(t)||_{0,\infty} \leq Ch^2\left(||u_0||_{3,\infty}^2 + \int_0^t ||u_t||_{3,\infty}^2 ds\right)^{1/2} \qquad (3.13)$$
$$+C\Delta t\left(\int_0^{t_n} (||u_{tt}||^2 + ||u_t||_1^2)ds\right)^{1/2},$$

$$||I_h u^n - u_h^n||_{1,\infty} = Ch^2(\log\frac{1}{h}) \quad \text{if} \quad u, u_t \in C^3. \qquad (3.14)$$

Proof: The proof is similar to the standard finite element arguments [5, 14, 15] using the results obtained in the previous sections, so we omit the details.

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