Finite Volume Methods on Voronoi Meshes

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Abstract

Two cell-centered finite difference schemes on Voronoi meshes are derived and investigated. Stability and error estimates in a discrete H^1-norm for both symmetric and nonsymmetric problems, including convection dominated, are proven. The theoretical results are illustrated with several numerical experiments.

Key words; cell-centered finite differences, upwind approximations, error estimates

AMS subject classification: 65N06, 65N12, 65N15

1 Introduction

In this paper we construct cell-centered finite volume difference schemes on Voronoi meshes for the following boundary value problem:

Find a function $u(\mathrm{x})$ that satisfies the differential equation and the boundary condition:

$$\mathcal{L}u(\mathrm{x}) = f(\mathrm{x}) \quad \text{in } \Omega \tag{1a}$$

$$u(\mathrm{x}) = 0 \qquad \text{on } \partial\Omega \tag{1b}$$

where

$$\mathcal{L}u = \nabla.(-a(\mathrm{x})\nabla u + \mathbf{b}(\mathrm{x})u)\,.$$

The domain Ω is a open subset of $\mathbb{R}^d$, $d = 2$ or 3.

We consistently use the notion of flux $\mathbf{q}$ defined below in (2a). We can rewrite the equation (1a) in the "flux" form

$$\mathbf{q} = -a\nabla u + \mathbf{b}u\,, \tag{2a}$$

$$\nabla.\mathbf{q} = f\,. \tag{2b}$$

Note that we can write the equation (2b) as

$$\int_{\partial V} (\mathbf{q}, \mathrm{n})\, ds = \int_V f\, d\mathrm{x} \tag{3}$$

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using Gauss formulae for sufficiently smooth subdomains $V \subset \Omega$.

The common approach of all finite volume methods is to approximate the integral of the normal component of the flux $\mathbf{q}$ and the source/sink function f on a set of finite (control) volumes V_i, $i = 1, \ldots, n$

$$\int_{\partial V_i} (\mathbf{q}, \mathrm{n})\, ds \approx \sum_{j \in \Sigma(i)} q_{ij}\,, \qquad \int_{V_i} f\, d\mathrm{x} \approx \phi_i\,,$$

and the equations (3) on V_i are replaced by the discrete equations

$$\sum_{j \in \Sigma(i)} q_{ij} = \phi_i\,, \quad i = 1, \ldots, n\,, \tag{4}$$

where the discrete fluxes q_{ij} are related to the approximation of u in some way.

This approach is very natural for many applications, as heat transfer, fluid flow in porous media, etc., where the differential problem (1) is derived from a conservation law in an integral form for the flux $\mathbf{q}$ like (3) and a constitutive relation between the flux $\mathbf{q}$ and the scalar variable u like (2a) is imposed.

Under some assumptions (see §2) the solution of the problem (1) satisfies an appropriate stability estimate, the maximum principle, and the operator $\mathcal{L}$ (or the corresponding bilinear form) is coercive. We would like to construct a family of stable (even for very small diffusion) numerical methods that inherit such properties as conservation, symmetry (the case $\mathbf{b} = 0$), maximum principle and coercivity in some discrete form. Moreover, we want our discretizations to work for general domains and general grids introduced into them.

It is natural to ask whether a finite volume method with good approximation properties that satisfies all conditions stated above can exist. We still do not know the answer of this question. Kershaw [17] has shown with simple argument that there is no nine point finite difference scheme on distorted quadrilateral meshes for the diffusion problem ($\mathbf{b} = 0$) that produces symmetric positive definite monotone matrix and has truncation error of second order. Recall that stability and truncation error of second order would imply second order convergence, but this is not a necessary condition (cf. [36], [18] and [23]).

Mixed finite element methods are conservative and have good approximation properties and this is the reason they are very active area of research. Russell et. al. [27] have proposed a conservative control volume mixed method that produces a nonsymmetric matrix for a symmetric differential problem. Thomas has developed mixed finite volume methods [34, 35] that also generates a non-symmetric matrix. Recently Arbogast, Wheeler and Yotov [2] have generalized the results in [38] for diffusion problems with tensor coefficients and have derived and analyzed new cell-centered finite volume difference schemes. We still do not know when or even whether these methods are monotone. The research in this direction has just begun [21].

Finite volume element methods are conservative, but not monotone in general [24, 7, 30]. Conservative finite volume difference methods on triangular meshes are well defined and monotone under quite restrictive conditions (acute-angled triangles [37, 15] , or constant coefficients [13]). Interesting results have been reported for quadrilateral vertex-centered finite volume difference schemes [25, 22, 33, 31]. but the consistent theory for such meshes is still not available.

We derive two schemes, central difference scheme (**CDS**) for symmetric or diffusion dominated problems and upwind difference scheme (**UDS**) for convection dominated problems and show that they are stable and first order accurate. These finite difference schemes are monotone (**CDS** only for symmetric problems), have the flexibility and the accuracy of the linear finite elements on simplexes (theoretically shown first order in a discrete H^1-norm and computationally observed second order in L^2-norm) and in addition they are conservative. Grids (Voronoi meshes) are not totally arbitrary, but general enough to cover all practical problems. The restriction on the grids seems quite natural since the discretization of the Laplacian with linear finite elements is monotone if and only if the triangulation is Delauney [16] - dual to a Voronoi mesh in a graph-theoretical sense.

Finite volume methods on Voronoi meshes are widely used in the engineering practice, especially for reservoir simulations [12, 26]. Our theory justifies their use and generalizes previous results for triangular meshes [37, 15].

The rest of the paper is organized in the following way. The properties of the continuous problem (1) are discussed in §2. The grids and control volumes are introduced in §3. In Section 4 we describe the discretization schemes and show that they satisfy the discrete maximum principle and the discrete operator is coercive. We prove the stability and error estimates in §5 and illustrate our theoretical results with a few numerical experiments gathered in Section 6.

2 Properties of the continuous problem

Here we write the precise form of the properties of the problem (1) and its solution mention above. We use the standard notation for Sobolev spaces [1]:

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \,:\, D^\alpha u \in L^p(\Omega), \text{ for } 0 \le |\alpha| \le m\}.$$

The norm in $W^{m,p}(\Omega)$ is denoted $\|.\|_{m,p,\Omega}$ and defined by

$$\|u\|_{m,p,\Omega} = \left(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|^p_{p,\Omega} \right)^{1/p}, \qquad 1 \le p < \infty\,,$$
$$\|u\|_{m,\infty,\Omega} = \max_{0\le|\alpha|\le m} \|D^\alpha u\|_{\infty,\Omega}\,, \qquad p = \infty$$

and $W^{m,2}(\Omega) = H^m(\Omega)$. The dual space $H^{-1}(\Omega)$ of $H^1_0(\Omega)$ is equipped with the norm

$$\|u\|_{-1,\Omega} = \sup_{\substack{v\in H^1_0(\Omega)\\ v\neq 0}} \frac{|<u,v>|}{\|v\|_{1,\Omega}}\,.$$

We also use spaces $H^s(\Omega)$ with real index $s > 0$ with norm defined in a consistent way (cf. [20], [1]).

We assume that the coefficients of the equation (1a) satisfy the conditions group for convenience in the following assumption.

Assumption 1 *Let the coefficient* $a(x) \in L^\infty(\Omega)$ *and* $\mathbf{b} \in \left(W^{d/2+\alpha,\infty}(\Omega)\right)^2$, $\alpha > 0$ *be such that*

$$(i) \quad a(x) \geq a_0 > 0\,, \tag{5a}$$

$$(ii) \quad \nabla.\mathbf{b} \geq 0\,. \tag{5b}$$

Under the Assumption 1, the problem (1) has a unique weak solution $u \in H_0^1(\Omega)$ for $f \in H^{-1}(\Omega)$ and u satisfies the the stability estimate

$$\|u\|_{1,\Omega} \leq C_1 \|f\|_{-1,\Omega}\,. \tag{CP1}$$

The operator $\mathcal{L}$ is coercive in $H_0^1(\Omega)$, i.e.,

$$(\mathcal{L}v, v) \geq C_2 \|v\|_{1,\Omega}^2 \quad \forall\, v \in H_0^1(\Omega)\,, \tag{CP2}$$

where

$$(\mathcal{L}u, v) = \int_\Omega a \nabla u \nabla v\, d\mathrm{x} - \int_\Omega (\mathbf{b}, \nabla v) u\, d\mathrm{x}\,.$$

Theorem 1 (Maximum principle) *Let the function* $u \in H^1(\Omega)$ *satisfies the inequality* $(\mathcal{L}u, v) \geq 0\, \forall\, v \in C_0^1(\Omega)\,, v \geq 0$ *and the Assumption 1 be fulfilled. Then if* $u(x) \geq k$ *on* $\partial\Omega$

$$\operatorname*{ess\,inf}_{x \in \Omega} u(x) \geq \min(0, k)\,.$$

We can write the maximum principle in slightly simplified and schematical way as

$$u \geq 0 \text{ on } \partial\Omega,\ \mathcal{L}u \geq 0 \text{ in } \Omega \ \text{ implies } \ u \geq 0 \text{ in } \Omega\,. \tag{CP3}$$

And finally we say that we have conservation property if for any connected volume $V = \cup_{i=1}^m V_i$ the following relation holds:

$$\int_{\partial V} (\mathbf{q}, \mathbf{n})\, ds = \sum_{i=1}^m \int_{\partial V_i} (\mathbf{q}, \mathbf{n})\, ds = \sum_{i=1}^m \int_{V_i} f\, d\mathrm{x} = \int_V f\, d\mathrm{x}\,. \tag{CP4}$$

We use the abbreviations (CP) to denote *continuous property.*

3 Grids, control volumes and discrete norms

We suppose that a grid $\overline{\omega} = \{\mathrm{x}_i\}_{i=1}^n$, $\mathrm{x}_i = (x_{i,1}, \ldots, x_{i,d})$ is introduced in the domain Ω and also Ω is partition into convex open polygons $\{V_i\}_{i=1}^n$, called control (finite) volumes, with the properties

$$\overline{\Omega} = \bigcup_{i=1}^n \overline{V}_i\,, \quad V_i \cap V_j = \emptyset \quad \text{and} \quad \overline{V}_i \cap \overline{V}_j = \gamma_{ij}\,, i \neq j\,,$$

where γ_{ij} is an interval in 2-D and convex polygon in 3-D. We denote the interior grid points with ω, i.e., $\omega = \overline{\omega} \cap \Omega$ and the grid points on $\partial\Omega$ with $\partial\omega$. The grid $\overline{\omega}$ and the control volumes $\{V_i\}_{i=1}^{n}$ are chosen in a such way that inside every V_i there is only one point x_i from $\overline{\omega}$. Later we will discuss how we can construct V_i given x_i and vice versa.

We use the standard symbols

$$h_i = diam(V_i)\,, \qquad h = \max_i h_i\,.$$

Functions defined for $\mathrm{x} \in \overline{\omega}$ are called grid functions. To emphasize their dependence of the triangulation we use the subscript h,

We use the notations:

$$\mathrm{m}(A) = \int_A d\mathrm{x}\,, \quad A \text{ is a measurable set of } \mathbb{R}^m,\ m = 1,2,3\,,$$

$$d(\mathrm{x},\mathrm{y}) = \left[\sum_{i=1}^{d} (x_i - y_i)^2\right]^{1/2}\,, \quad d(\mathrm{x},\Omega) = \inf_{\mathrm{y}\in\Omega} d(\mathrm{x},\mathrm{y})\,,$$

$$\Sigma(i) = \{j\,:\, j \neq i \text{ and } \overline{V}_i \cap \overline{V}_j \neq \emptyset\}\,.$$

Given the grid functions $u_h(\mathrm{x})$, $v_h(\mathrm{x})$, $x \in \overline{\omega}$ we define the following discrete inner products and norms:

$$(u_h, v_h) = \sum_{\mathrm{x}_i \in \omega} \mathrm{m}(V_i) u_h(\mathrm{x}_i) v_h(\mathrm{x}_i)\,, \quad \|u_h\|^2_{0,\omega} = (u_h, u_h);$$

$$|u_h|^2_{1,\omega} = \frac{1}{2} \sum_{\mathrm{x}_i \in \overline{\omega}} \sum_{j \in \Sigma(i)} \mathrm{m}(\gamma_{ij}) d(x_i, x_j) \left(\frac{u_h(\mathrm{x}_i) - u_h(\mathrm{x}_j)}{d(\mathrm{x}_i, \mathrm{x}_j)}\right)^2\,.$$

The discrete H^1–norm is defined by

$$\|u_h\|^2_{1,\omega} = \|u_h\|^2_{0,\omega} + |u_h|^2_{1,\omega}\,.$$

We also need the negative norm:

$$\|u_h\|_{-1,\omega} = \sup_{v_h \neq 0} \frac{|(u_h, v_h)|}{\|v_h\|_{1,\omega}}\,.$$

We consider Voronoi control volumes, i.e., the control volume V_i is the set of points in Ω closest to x_i. It is easy to see that these control volumes are convex polygons. By construction the interval with end points x_i and x_j is perpendicular to γ_{ij} and the plane where γ_{ij} lies intersects this interval in the middle point denoted with x_{ij}. Voronoi volumes are easily constructed from the mesh $\overline{\omega}$. We refer for discussion of geometrical properties of Voronoi volumes to the survey papers by Aurenhammer [3] and Fortune [10]. We show a fragment of Voronoi triangulation on Fig. 1. We will call such grids *Voronoi cell-centered* grids.

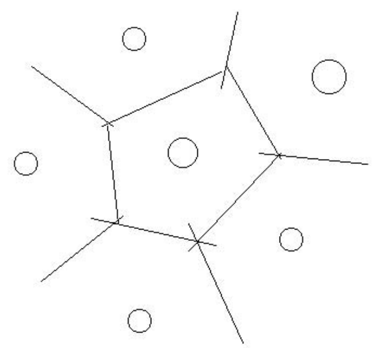

Figure 1: Voronoi triangulation

The planar Voronoi triangulation and the Delaunay triangulation are dual in a graph-theoretical sense. Delaunay triangulation consists from triangles with properties that the circumscribed circle of each triangle does not contain vertices of any other triangles. Given Delaunay triangulation, i.e., a grid $\overline{\omega}$ it is easy to construct Voronoi volumes. We note that the Delaunay triangulation is one of the most popular in computational mesh generation [11] especially for finite element computations because of its optimal properties [5, 32].

As an example of a mesh generated from the control volumes we consider circumscribed triangulations, i.e., there exists a circumscribed circle around each control volume and the center of the circle is inside the control volume. The grid consists of the centers of the circumscribed circles. A fragment of such triangulation is depicted on Fig. 2. Circumscribed triangulations are difficult to construct in general domains Ω and we will not consider such meshes, but we like to note that the theory presented in this paper cover also such triangulatons.

Given n points in $\mathbb{R}^d$, $d = 2, 3$ we can construct a Voronoi cell-centered mesh. This includes many "pathological" cases. In order to develop reasonable theory we impose some regularity conditions on the mesh. The sign $\sim$ is used to simplify the notation and $a \sim b$ means that there are two positive constants C_1 and C_2 independent from a and b such that $C_1 a \le b \le C_2 a$.

Assumption 2 (FV regular triangulations) *We say that a cell–centered triangulation* $\{V_i\}_{i=1}^n$ *is* finite volume regular *if every control volume satisfies*

$$m(V_i) \sim h_i^d\,, \tag{6a}$$

$$d(x_i, x_j) \sim h_k \quad k = i \text{ or } j\,, \tag{6b}$$

$$d(x_{ij}, \gamma_{ij}) \sim h_k \quad k = i \text{ or } j\,. \tag{6c}$$

Remark 1 (6a) is the standard assumption for quasi-uniform finite element meshes (cf. Ciarlet [8, p. 132]). The second assumption (6b) is assumed in

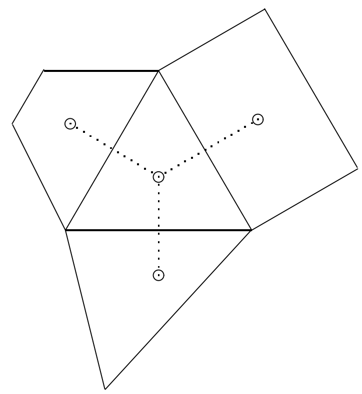

Figure 2: Circumscribed triangulation

order to prevent two points to be too close and eventual extra ill-conditioning of the discrete scheme resulting from this. In finite element triangulations this means all angles are bounded from below with $\alpha > 0$. The third assumption (6c) guarantees that x_{ij} is not too far from γ_{ij} and is necessary for the approximation properties of the methods (see the discussion in Lemma 3). The last assumption means in finite element context that all angles are bounded from above with $\pi - \alpha$. For finite element triangulations in 2-D (6a) implies (6b) and (6c). We do not know whether this is also true for Voronoi cell-centered meshes. Note that we consider both 2 and 3-D meshes.

4 Discretization schemes

In this section we define the expressions of the discrete fluxes q_{ij} through the values of the approximation u_h of the continuous solution u. When these expressions are substituted into the equations (4) we get a system of linear equations

$$\mathcal{L}_h u_h = \phi\,. \tag{7}$$

We will investigate the properties of the discrete operator $\mathcal{L}_h$ and compare them to the properties of $\mathcal{L}$.

We approximate the scaled by $\mathrm{m}(V_i)$ equation (3)

$$\frac{1}{\mathrm{m}(V_i)}\int_{\partial V_i}(-a\nabla u + \mathbf{b}u, \mathrm{n})\,ds = \frac{1}{\mathrm{m}(V_i)}\int_{V_i} f(x)\,d\mathrm{x}\,. \tag{8}$$

Denote

$$\mathbf{w} = -a(\mathrm{x})\nabla u(\mathrm{x}) \quad \text{ and } \quad \mathbf{v} = \mathbf{b}(\mathrm{x})u(\mathrm{x}).$$

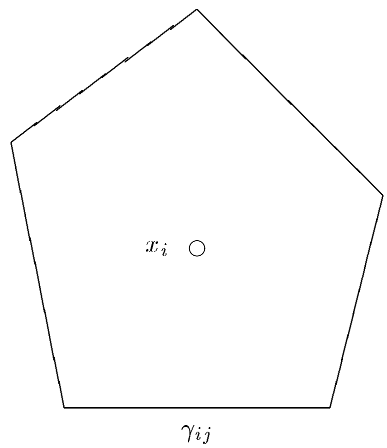

Figure 3: General control volume V_i

Splitting $\partial V_i = \cup_{j\in\Sigma(i)}\gamma_{ij}$ (see Fig. 3) the left-hand side of this identity is written in the form:

$$\frac{1}{\mathrm{m}(V_i)}\left[\int_{\partial V_i}(\mathbf{w},\mathbf{n})\,ds + \int_{\partial V_i}(\mathbf{v},\mathbf{n})\,ds\right] =$$
$$\frac{1}{\mathrm{m}(V_i)}\left[\sum_{j\in\Sigma(i)}\int_{\gamma_{ij}}(\mathbf{w},\mathbf{n})\,ds + \sum_{j\in\Sigma(i)}\int_{\gamma_{ij}}(\mathbf{v},\mathbf{n})\,ds\right] \quad (9)$$

We split the approximation of the balance equation (9) in two parts

$$\mathcal{L}_h u_h = \mathcal{L}_h^{(2)} u_h + \mathcal{L}_h^{(1)} u_h \quad (10)$$

where $\mathcal{L}_h^{(2)}$ is the part arising from the approximation of the second derivatives, and $\mathcal{L}_h^{(1)}$ comes from the approximation of the first derivatives. We have the expressions

$$\begin{aligned} \mathcal{L}_h^{(2)} u_h &= \sum_{j\in\Sigma(i)} w_{i,j}\,, \quad x_i \in \omega\,, \\ \mathcal{L}_h^{(1)} u_h &= \sum_{j\in\Sigma(i)} v_{i,j}\,, \quad x_i \in \omega\,. \end{aligned} \quad (11)$$

In these formulas w_{ij} and v_{ij} are some approximations of the corresponding integrals $\int_{\gamma_{ij}}(\mathbf{w},\mathbf{n})\,ds$ and $\int_{\gamma_{ij}}(\mathbf{v},\mathbf{n})\,ds$. Now, in order to complete the finite difference scheme we have to express the approximate fluxes w_{ij} and v_{ij} by the approximate values $u_h(\mathrm{x})$ of the solution $u(\mathrm{x})$ at the grid points.

We denote by $\beta_{i,j}$ an approximation of the integral $\int_{\gamma_{ij}} (\mathbf{b}, \mathbf{n})\, ds$ with the properties:

$$(i) \quad \beta_{i,j} + \beta_{j,i} = 0\,. \tag{12a}$$

$$(ii) \quad |\beta_{i,j}| \leq C\, \mathrm{m}(\gamma_{ij}) \|\mathbf{b}\|_{d/2+\alpha,\infty,\Omega}\,, \tag{12b}$$

$$(iii) \quad \left| \int_{\gamma_{ij}} (\mathbf{b}, \mathbf{n})\; ds - \beta_{i,j} \right| \leq C h^{d+\alpha} |\mathbf{b}|_{d/2+\alpha,\infty,\Omega}\,, \tag{12c}$$

where C is a positive constant and $\alpha > 0$. Similar conditions for β are used by Baba and Tabata in constructing upwind finite element methods for parabolic problems [4].

We consider some examples of quadrature formulas that satisfy conditions (12). Let γ_{ij} be an interval with end points a_1 and a_2 and a middle point a_{12}. The following well known quadrature formulas (2-D triangulations) clearly satisfy the conditions (12)

$$\beta_{ij} = (\mathbf{b}, \mathbf{n})(a_{12})\, \mathrm{m}(\gamma_{ij}),$$

$$\beta_{ij} = \frac{\mathrm{m}(\gamma_{ij})}{2} \left[(\mathbf{b}, \mathbf{n})(a_1) + (\mathbf{b}, \mathbf{n})(a_2) \right].$$

For triangular and rectangular faces γ_{ij} (3-D triangulations) the quadrature formulae

$$\beta_{ij} = (\mathbf{b}, \mathbf{n})(a_{bary})\, \mathrm{m}(\gamma_{ij})$$

fulfills (12) with a_{bary} the barycenter of γ_{ij}.

We consider the following approximations.

4.1 Central difference scheme (CDS)

We call this scheme "central" because of the analogy of $\mathcal{L}_h^{(1)}$ and a central difference approximation of the first derivatives. The approximate fluxes are defined by:

$$w_{ij}(\mathrm{x}) = -\frac{\mathrm{m}(\gamma_{ij})}{\mathrm{m}(V_i)} k_{ij} \frac{[u_{h,j} - u_{h,i}]}{d(x_i, x_j)}\,, \tag{13a}$$

$$v_{ij}(\mathrm{x}) = \frac{\beta_{i,j}}{\mathrm{m}(V_i)} \left[\frac{d(x_j, x_{ij})}{d(x_i, x_j)} u_{h,i} + \frac{d(x_i, x_{ij})}{d(x_i, x_j)} u_{h,j} \right] \tag{13b}$$

where

$$k_{i,j} = \left(\frac{1}{d(x_i, x_j)} \int_{x_i}^{x_j} \frac{ds}{a(s)} \right)^{-1}.$$

is the harmonic average of diffusion coefficient.

An application of the discrete maximum principle shows that **CDS** is stable if the following inequalities are satisfied

$$P_i = \max_{j\in\Sigma(i)} \frac{|\beta_{i,j}|}{\mathrm{m}(\gamma_{ij})}.\frac{d(x_i,x_{ij})}{k_{i,j}} \leq 1\,, \qquad x_i = 1,\dots,n\,. \tag{14}$$

In some application P_i is called a local (cell) Peclet number (cf. [14], [28]). Note that the quantity $|\beta_{i,j}|/\,\mathrm{m}(\gamma_{ij})$ does not depend upon h and therefore the inequalities (14) are satisfied only for sufficiently small h.

We will deduce the properties of **CDS** using the tools developed for **UDS**.

4.2 Upwind difference scheme (UDS)

One of the ways to find stable finite difference approximation for convection-diffusion boundary value problem is to use upwind approximation for the first derivatives. In this case, $\mathcal{L}^{(2)}$ is defined as in **CDS** and the terms $v_{i,j}$ in $\mathcal{L}^{(1)}$ are approximated in the following way:

$$v_{i,j} = \beta^+_{i,j} u_{h,i} + \beta^-_{i,j} u_{h,j} \tag{15a}$$

where $\beta^+_{i,j}$ and $\beta^-_{i,j}$ are defined via the formulas

$$\beta^+_{i,j} = \frac{1}{\mathrm{m}(V_i)}.\frac{(\beta_{i,j}+|\beta_{i,j}|)}{2}\,, \qquad \beta^-_{i,j} = \frac{1}{\mathrm{m}(V_i)}.\frac{(\beta_{i,j}-|\beta_{i,j}|)}{2}\,. \tag{15b}$$

The definition of the discrete fluxes w_{ij} and v_{ij} (13), (15) imply that

$$w_{ij} = -w_{ji} \quad \text{and} \quad v_{ij} = -v_{ji}\,. \tag{16}$$

We can easily show using (16) that the discrete flux through the boundary of any connected volume V such that $V = \cup_{i=1}^m V_i$ and V_i are control volumes, equals to the sum of sources/sinks

$$\sum_{\partial V} q_{ij} = \sum_{i=1}^m \sum_{j\in\Sigma(i)} q_{ij} = \sum_{i=1}^m \phi_i\,, \tag{DP4}$$

which is the discrete conservation law corresponding to (CP4). We use the abbreviations (DP) to denote *discrete property.*

In order to investigate the properties of the **UDS** we need the following auxiliary result.

Proposition 1 *Let* $\mathbf{b}(x) \in (W^{d/2+\alpha,\infty}(\Omega))^d$, $\alpha > 0$ *and there exists a positive constant* β_0 *such that*

$$\int_{\partial V} (\mathbf{b}(x),\mathbf{n})\,ds \geq \beta_0\,\mathrm{m}(V) \tag{17}$$

for any volume $V \subset \Omega$ with Lipschitz–continuous boundary $\partial V = \cup_{j\in\Sigma(i)}\gamma_{ij}$. Suppose that $\beta_{i,j}$ satisfies the condition (12c)*. Then there exists h_0 such that for $h \in (0, h_0)$ the following inequality holds:*

$$\sum_{j\in\Sigma(i)} \beta_{i,j} \geq c_0 \,\mathrm{m}(V)\,, \tag{18}$$

where $c_0 = \beta_0 - O(h^\alpha)$.

Proof: It follows from the FV regularity of the control volume V and the condition (12c). □

We replace the condition (5a) of Assumption 1 with the stronger one.

Assumption 3 $\mathbf{b}(x) \in \left(W^{d/2+\alpha,\infty}(\Omega)\right)^d$, $\alpha > 0$ *and* $\nabla.\mathbf{b}(x)) \geq \beta_0 > 0$ *for almost every $x \in \Omega$.*

Remark 2 We can consider the left hand side of (18) as a definition of the discrete divergence operator. Then the above proposition means that, if the divergence of the vector $\mathbf{b}$ is greater than $\beta_0 > 0$, the discrete analogy of $div(\mathbf{b})$ is also positive for sufficiently small h.

First we will prove that the considered scheme is monotone.

Proposition 2 *Let the Assumptions 1, 2 and 3 be satisfied, the discrete fluxes $w_{i,j}$ and $v_{i,j}$ be defined by the formulas* (13a) *and* (15)*, respectively, and the approximations $\beta_{i,j}$ fulfill the condition* (12c)*. Then* **UDS** *satisfies the discrete maximum principle and the corresponding matrix $\mathcal{L}_h$ is an* **M***–matrix.*

Proof: Let $a_{i,j}$ be the coefficients in front of $u_{h,j}$ in the i^{th} equation. Then it is enough to check the conditions [13]:

1. $a_{i,i} > 0 \qquad a_{i,j} \leq 0 \quad j \neq i;$
2. $a_{i,i} + \sum_{j\in\Sigma(i)} a_{i,j} > 0$, i.e., A is strictly diagonally dominant.

We have

1.

$$a_{i,i} = \frac{1}{\mathrm{m}(V_i)} \sum_{j\in\Sigma(i)} \left[\frac{\mathrm{m}(\gamma_{ij})}{d(x_i,x_j)} k_{i,j} + \beta_{i,j} + |\beta_{i,j}|\right] > 0\,,$$

$$a_{i,j} = \frac{1}{\mathrm{m}(V_i)} \left[-\frac{\mathrm{m}(\gamma_{ij})}{d(x_i,x_j)} k_{i,j} + \beta_{i,j} - |\beta_{i,j}|\right] < 0\,,$$

2.

$$a_{i,i} + \sum_{j\in\Sigma(i)} a_{i,j} = \frac{1}{\mathrm{m}(V_i)} \sum_{j\in\Sigma(i)} \beta_{i,j} \geq c_0 > 0\,.$$

The last inequality follows from the Proposition 1. $\square$

Similarly as (CP3) we can write the discrete maximum principle as

$$u_h \geq 0 \text{ on } \partial\omega,\ \mathcal{L}_h u \geq 0 \text{ in } \omega \ \text{ implies } \ u_h \geq 0 \text{ in } \omega\,. \tag{DP3}$$

Note that to prove Proposition 2 we used only that $k_{ij} > 0$, the Assumption 3 and (12c). As a consequence of Proposition 2 we obtain that the problem (7) has an unique solution. With more elaborated analysis we can prove Proposition 2 using only (6a). For symmetric problems ($\mathbf{b} = 0$) the only condition to have a monotone scheme is the condition (5a) of Assumption 1.

Now we show that the coercivity of the differential operator $\mathcal{L}$ is inherited by the discrete operator $\mathcal{L}_h$, i.e., $\mathcal{L}_h$ considered as a matrix is positive definite.

Proposition 3 *Let the Assumptions 1, 2 and 3 be satisfied, the discrete fluxes $w_{i,j}$ and $v_{i,j}$ be defined by the formulas* (13a) *and* (15)*, respectively, and the approximations $\beta_{i,j}$ fulfill the conditions* (12)*. Then the matrix $\mathcal{L}_h$ of* **UDS** *is a positive definite and there exists a constant C such that the following inequality is true:*

$$(\mathcal{L}_h y, y) \geq C\|y\|^2_{1,\omega}\ ,\ \textit{for all } y \in D^0 = \{y,\ y_{|\gamma} = 0\}. \tag{DP2}$$

The constant C depends only on the ratio $a(x)/|\mathbf{b}(x)|$.

Proof: Let $z(x)$ and $y(x)$ be grid functions from D^0. Then

$$\begin{aligned}(\mathcal{L}_h y, z) &= -\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \mathrm{m}(V_i) w_{i,j} z_j + \sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \mathrm{m}(V_i) v_{i,j} z_j \\ &= I + J\,.\end{aligned} \tag{19}$$

We transform the sums in formulae (19)

$$\begin{aligned} I &= -\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \mathrm{m}(V_i)\left[\frac{\mathrm{m}(\gamma_{ij})}{\mathrm{m}(V_i)} k_{i,j}\frac{[y_j - y_i]}{d(x_i,x_j)}\right] z_i \\ &= \frac{1}{2}\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \frac{\mathrm{m}(\gamma_{ij})}{d(x_i,x_j)} k_{i,j}\left([y_j - y_i]z_j - [y_j - y_i]z_i\right) \\ &= \frac{1}{2}\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \mathrm{m}(\gamma_{ij}) d(x_i,x_j) k_{i,j}\frac{[y_j - y_i]}{d(x_i,x_j)}\cdot\frac{[z_j - z_i]}{d(x_i,x_j)}\,. \end{aligned}$$

Using (15) we rewrite J in the following way

$$\begin{aligned} J &= \frac{1}{2}\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \left[(\beta_{i,j} + |\beta_{i,j}|)y_i + (\beta_{i,j} - |\beta_{i,j}|)y_j\right] \\ &= \frac{1}{2}\sum_{x_i\in\omega}\left[\sum_{j\in\Sigma(i)} \beta_{i,j} y_i z_i + \sum_{j\in\Sigma(i)} |\beta_{i,j}|(y_i - y_j) + \sum_{j\in\Sigma(i)} \beta_{i,j} y_j z_i\right] \\ &= J_1 + J_2 + J_3\,. \end{aligned} \tag{20}$$

We now transform the second term in (20)

$$J_2 = \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in \Sigma(i)} |\beta_{i,j}|(y_i - y_j) z_i = \frac{1}{4} \sum_{x_i \in \omega} \sum_{j \in \Sigma(i)} |\beta_{i,j}|(y_i - y_j)(z_i - z_j)$$

and the third term in (20)

$$\begin{aligned} J_3 = \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in \Sigma(i)} \beta_{i,j} y_j z_i &= \frac{1}{4} \sum_{x_i \in \omega} \sum_{j \in \Sigma(i)} \beta_{i,j} y_j z_i + \beta_{j,i} y_i z_j \\ &= \frac{1}{4} \sum_{x_i \in \omega} \sum_{j \in \Sigma(i)} \beta_{i,j} (y_j z_i - y_i z_j) \,. \end{aligned}$$

Finally we get

$$\begin{aligned} (\mathcal{L}_h y, z) \quad = \quad & \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in \Sigma(i)} \mathrm{m}(\gamma_{ij}) d(x_i, x_j) k_{i,j} \frac{[y_j - y_i]}{d(x_i, x_j)} \cdot \frac{[z_j - z_i]}{d(x_i, x_j)} \\ & + \frac{1}{2} \sum_{x_i \in \omega} \left(\sum_{j \in \Sigma(i)} \beta_{i,j} \right) y_i z_i + \frac{1}{4} \sum_{x_i \in \omega} \sum_{j \in \Sigma(i)} |\beta_{i,j}|(y_i - y_j)(z_i - z_j) \\ & + \frac{1}{4} \sum_{x_i \in \omega} \sum_{j \in \Sigma(i)} \beta_{i,j} (y_j z_i - y_i z_j) \,. \end{aligned}$$

Letting $z = y$ in the above formula the desired result follows using Proposition 1 and the FV regularity of the control volumes. $\Box$ If is easy to show that (DP2) is also satisfied for **CDS** is the problem (1) is diffusion dominated.

5 Stability and error estimates

The stability of problem (7) is a simple consequence of the positive definiteness of the matrix L. Namely, we prove the following lemma.

Lemma 1 *Let the Assumptions 2 and 3 be satisfied. Then for* **UDS** *the following a priori estimate is valid:*

$$\|u_h\|_{1,\omega} \le C \|\phi\|_{-1,\omega} \,, \tag{DP1}$$

where u_h is the discrete solution and ϕ is the right-hand side of (7). The constant C in this estimate does not depend on h or ϕ.

The error analysis presented here is done in the general framework of the methods developed in [29] and [9]. We consider only the case when $a(x) \equiv 1$. Let

$$z(x) = u_h(x) - u(x) \,, \; x \in \omega$$

be the error of the finite difference method. Substituting $u_h = z + u$ in (7) we obtain

$$\mathcal{L}_h z = \phi - \mathcal{L}_h u \equiv \psi\,. \tag{21}$$

Then using (7)–(15) we transform ψ in the following form

$$\sum_{j\in\Sigma(i)} \left[\frac{1}{\mathrm{m}(V_i)}\int_{\gamma_{ij}} (\mathbf{w},\mathbf{n})\,ds - w_{i,j}\right] + \sum_{j\in\Sigma(i)} \left[\frac{1}{\mathrm{m}(V_i)}\int_{\gamma_{ij}} (\mathbf{v},\mathbf{n})\,ds - v_{i,j}\right] \equiv \psi_{1,i} + \psi_{2,i} = \psi_i\,.$$

We define the local truncation error in the following way:

$$\eta_{i,j} = \frac{1}{\mathrm{m}(\gamma_{ij})}\int_{\gamma_{ij}} (\mathbf{w},\mathbf{n})\,ds - \frac{\mathrm{m}(V_i)}{\mathrm{m}(\gamma_{ij})} w_{i,j}\,, \tag{22a}$$

$$\mu_{i,j} = \frac{1}{\mathrm{m}(\gamma_{ij})}\int_{\gamma_{ij}} (\mathbf{v},\mathbf{n})\,ds - \frac{\mathrm{m}(V_i)}{\mathrm{m}(\gamma_{ij})} v_{i,j}\,. \tag{22b}$$

First we consider the term (ϕ_2, z). By the definition of the discrete inner product and $\phi_{2,i}$ we have

$$\begin{aligned}
(\phi_2, z) &= \sum_{x_i\in\omega} \mathrm{m}(V_i)\phi_{2,i} z_i \\
&= \sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \left[\int_{\gamma_{ij}} \mathbf{w},\mathbf{n})\,ds + \mathrm{m}(\gamma_{ij})k_{ij}\frac{[u_j - u_i]}{d(x_i,x_j)}\right] z_i\,.
\end{aligned}$$

We can regroup the terms (we call this nonuniform summation by parts) to get

$$\begin{aligned}
(\phi_2, z) &= \frac{1}{2}\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \left\{\left[\int_{\gamma_{ij}} (\mathbf{w},\mathbf{n}) + \mathrm{m}(\gamma_{ij})k_{ij}\frac{[u_j - u_i]}{d(x_i,x_j)}\right] z_i \right.\\
&\qquad\qquad \left. + \left[\int_{\gamma_{ji}} (\mathbf{w},\mathbf{n}) + \mathrm{m}(\gamma_{ji})k_{ji}\frac{[u_i - u_j]}{d(x_i,x_j)}\right] z_j\right\} \\
&= -\frac{1}{2}\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \left[\int_{\gamma_{ij}} (\mathbf{w},\mathbf{n}) + \mathrm{m}(\gamma_{ij})k_{ij}\frac{[u_j - u_i]}{d(x_i,x_j)}\right] [z_j - z_i] \\
&= -\frac{1}{2}\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} d(x_i,x_j)\,\mathrm{m}(\gamma_{ij}) \\
&\qquad\qquad \left[\frac{1}{\mathrm{m}(\gamma_{ij})}\int_{\gamma_{ij}} (\mathbf{w},\mathbf{n}) - \frac{\mathrm{m}(V_i)}{\mathrm{m}(\gamma_{ij})} w_{i,j}\right] \frac{[z_j - z_i]}{d(x_i,x_j)} \\
&= -\frac{1}{2}\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} d(x_i,x_j)\,\mathrm{m}(\gamma_{ij})\eta_{i,j}\frac{[z_j - z_i]}{d(x_i,x_j)}\,.
\end{aligned}$$

By the Cauchy–Schwartz inequality follows

$$\begin{aligned}(\phi_2, z) &\leq \left(\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \mathrm{m}(\gamma_{ij}) d(x_i,x_j)\eta_{i,j}^2\right)^{1/2}\left(\sum_{x_i\in\omega}\sum_{j\in\Sigma(i)} \mathrm{m}(\gamma_{ij})\frac{[z_j-z_i]^2}{d(x_i,x_j)}\right)^{1/2}\\ &\leq \|\eta\|_{*,\omega}\|z\|_{1,\omega}\,.\end{aligned}$$

Here for convenience we denote with $\|\eta\|_{*,\omega}$ the first sum above.

Likewise

$$(\phi_1, z) \leq \|\mu\|_{*,\omega}\|z\|_{1,\omega}\,.$$

Summarizing these results and using Proposition 3 we obtain the following main result.

Lemma 2 *Let the Assumptions 1, 2 and 3 be satisfied. The error* $z(x) = u_h(x) - u(x)$, $x \in \omega$ *of* **UDS** *satisfies the a priori estimate*

$$\|z\|_{1,\omega} \leq C\left(\|\eta\|_{*,\omega} + \|\mu\|_{*,\omega}\right) \tag{23}$$

where the components $\eta_{i,j}$ *and* $\mu_{i,j}$ *of the local truncation error are defined by (22) with approximate fluxes* $w_{i,j}$ *and* $v_{i,j}$ *determined by (13a), (15). The constant* C *in this estimate does not depend on* h *or* z.

In order to use the estimate (23) of Lemma 2 we have to bound the corresponding norms of the local truncation error components $\eta_{i,j}$ and $\mu_{i,j}$ defined by (22). These estimates are provided in the lemmas given below.

Consider one fixed face γ_{ij} and the prism e_{ij} with two faces through x_i and other faces are parallel to the straight line (x_i, x_j) and go through the boundary of γ_{ij}. Note that γ_{ij} is a convex polygon by construction.

Lemma 3 *Let the solution of the problem (1) be* H^s*-regular,* $\frac{3}{2} < s$*, and the component of the local truncation error* $\eta_{i,j}$ *be defined by (22a) with the approximate flux* $w_{i,j}$ *determined by (13a). Then the following estimate holds:*

$$|\eta_{i,j}| \leq Ch^{s-d/2-1}|u|_{s,e_{ij}}\,,\ \frac{3}{2} < s \leq 2. \tag{24}$$

Proof: Consider the component $\eta_{i,j}(u)$ for the **UDS**. Then

$$\begin{aligned}\eta_{i,j}(u) &= -\frac{\mathrm{m}(V_i)}{\mathrm{m}(\gamma_{ij})}w_{i,j} + \frac{1}{\mathrm{m}(\gamma_{ij})}\int_{\gamma_{ij}} (\mathbf{W}, \mathbf{n})\, ds\\ &= \frac{[u_j - u_i]}{d(x_i,x_j)} - \frac{1}{\mathrm{m}(\gamma_{ij})}\int_{\gamma_{ij}} \frac{\partial u}{\partial n}\, ds\end{aligned}$$

First we assume that $x_{ij} \in \gamma_{ij}$. Using the imbedding of Sobolev spaces $H^s(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$, $\alpha = s - d/2 > 0$ and $H^s(\Omega) \hookrightarrow H^\chi(\gamma_{ij})$, $\chi = s - 1/2 > 1$ (cf. [1]) we conclude that $\eta(u)$ is a bounded linear functional in $H^s(e_{ij})$, $3/2 < s$. It is easy to check that $\eta_{i,j}(u)$ vanishes if u is a polynomial of first degree. Therefore, by the Bramble-Hilbert lemma argument [6] we get that

$$|\eta_{i,j}(u)| \le Ch^{s-d/2-1}|u|_{s,e_{ij}}\,, \quad \frac{3}{2} < s \le 2\,. \tag{25}$$

Next we consider the case when x_{ij} is not in γ_{ij} for a FV regular mesh, i.e., the Assumption 2 and in particular (6c) is satisfied. Denote with γ a translation of γ_{ij} such that $x_{ij} \in \gamma$. It is easy to see that

$$\left| \int_{\gamma_{ij}} \frac{\partial u}{\partial n}\, ds - \int_{\gamma} \frac{\partial u}{\partial n}\, ds \right| \le h^{d-1}\|u\|_{s,\hat{e}_{ij}}\,, \tag{26}$$

where $m(e_{ij}) \le Cm(\hat{e}_{ij})$ and the constant C does not depend on h or e_{ij}. The contribution to the error of the term (26) is at most of order h, so we neglect it. $\square$

Lemma 4 *Let the solution of the problem (1) be H^s-regular, $\frac{3}{2} < s$, and the component of the local truncation error $\mu_{i,j}$ be defined by (22b) with the approximate flux $v_{i,j}$ determined by* (13b) *and* (15). *Then the following estimate holds:*

$$|\mu_{i,j}| \le \left\{ \begin{array}{ll} Ch^{s-d/2}\|\mathbf{b}\|_{d/2+\alpha,\infty,\Omega}\|u\|_{s,e_{ij}} & \text{for } \mathbf{CDS}, \\ Ch^{1-d/2}\|\mathbf{b}\|_{d/2+\alpha,\infty,\Omega}\|u\|_{s,e_{ij}} & \text{for } \mathbf{UDS}, \end{array} \right. \tag{27}$$

where $\frac{d}{2} < s \le 2$.

Proof: We consider two cases. For **UDS** suppose $\beta_{i,j} > 0$. Then

$$\mu_{i,j}(u) = \frac{1}{\mathrm{m}(\gamma_{ij})} \int_{\gamma_{ij}} (\mathbf{b}, \mathbf{n}) u\, ds - \frac{1}{\mathrm{m}(\gamma_{ij})} \beta_{i,j} u_i. \tag{28}$$

After the scaling is performed, the truncation error $\mu_{i,j}$ (28) simplifies to

$$\mu_{i,j}(\bar{u}) = \int_{\bar{\gamma}_{ij}} (\bar{\mathbf{b}}, \mathbf{n})\bar{u}\, ds - \bar{\beta}_{i,j}\bar{u}_i.$$

Denote $l_1(\bar{\mathbf{b}}, \bar{u}) = -\mu_{i,j}(\bar{u})$. We represent l_1 in the following way:

$$l_1(\bar{\mathbf{b}}, \bar{u}) = \bar{\beta}_{i,j} p_1(\bar{u}) + c(\bar{\mathbf{b}}, \bar{u}) + \bar{u}_i q(\mathbf{b})\,,$$

where the linear functionals $p_1(\bar{u})$, $q(\mathbf{b})$ and the bilinear functional $c(\bar{\mathbf{b}}, \bar{u})$ are defined by

$$\begin{aligned} p_1(\bar{u}) &= \bar{u}_i - \int_{\bar{\gamma}_{ij}} \bar{u}\, ds, \\ c(\bar{\mathbf{b}}, \bar{u}) &= \int_{\bar{\gamma}_{ij}} [\bar{\beta}_{i,j} - (\bar{\mathbf{b}}, \mathbf{n})][\bar{u} - \bar{u}_i]\, ds, \\ q(\mathbf{b}) &= \int_{\bar{\gamma}_{ij}} [\beta_{i,j} - (\mathbf{b}, \mathbf{n})]\, ds\,. \end{aligned}$$

It is easy to see that $p_1(\bar{u})$ is bounded for $\bar{u} \in H^s(\bar{e}_{ij})$ and vanishes for constants. Using a variation of Bramble-Hilbert lemma argument (see [13]) we get

$$|p_1(u)| \le Ch^{1-d/2}(|u|_{1,e_{ij}} + h^{s-1}|\bar{u}|_{s,\bar{e}_{ij}}), \quad \frac{d}{2} < s \le 2.$$

Obviously $c(\bar{\mathbf{b}}, \bar{u})$ is a bilinear functional bounded for $(\bar{\mathbf{b}}, \bar{u}) \in \left(W^{1,\infty}(\bar{e}_{ij})\right)^d \times H^1(\bar{e}_{ij})$ and vanishes for r, s polynomials of zero degree, i.e., $c(r, \bar{u}) = 0$ for $\bar{u} \in H^1(\bar{e}_{ij})$ and $c(\bar{\mathbf{b}}, s) = 0$ for $\bar{\mathbf{b}} \in (W^{1,\infty}(\bar{e}_{ij}))^d$. Then by the bilinear variant of the Bramble-Hilbert lemma (see [8]) we have

$$|c(\mathbf{b}, u)| \le Ch^{2-d/2}|\mathbf{b}|_{1,\infty,e_{ij}}|u|_{1,e_{ij}}.$$

And finally the linear functional is estimated by the assumption (12c)

$$|q(\mathbf{b})| \le Ch^{d+\alpha}\|\mathbf{b}\|_{d/2+\alpha,\infty,\Omega}.$$

Combining the estimates for $p(.)$, $c(.,.)$ and $q(.)$ we get the first part of (27).

Similarly for **CDS** we define $l_2(\bar{\mathbf{b}}, \bar{u}) = -\mu_{i,j}(\bar{u})$ and

$$l_2(\bar{\mathbf{b}}, \bar{u}) = \bar{\beta}_{i,j} p_2(\bar{u}) + c(\bar{\mathbf{b}}, \bar{u}) + \bar{u}_i q(\mathbf{b}),$$

where the linear functional $p_2(.)$ is given by the formulae

$$p_2(\bar{u}) = [\alpha_j \bar{u}_i + \alpha_j \bar{u}_j] - \int_{\bar{\gamma}_{ij}} \bar{u}\, ds$$

and $q(.)$ and $c(.,.)$ are the same as above.

$p_2(\bar{u})$ is bounded for $\bar{u} \in H^s(\bar{e}_{ij})$, $\frac{d}{2} < s$ and vanishes for all polynomials of first degree. Hence

$$|p_2(u)| \le Ch^{s-d/2}|u|_{s,e_{ij}}, \quad \frac{d}{2} < s \le 2.$$

□

Now we are ready to prove the main result of this section.

Theorem 2 *If the solution $u(x)$ of the problem (1) is H^s-regular, with $\frac{3}{2} < s \le 2$ and the Assumptions 1, 2 and 3 are satisfied then* **UDS** *and* **CDS** *(when is stable) have $O(h^{s-1})$ rate of convergence in the H^1– discrete norm, i.e.,*

$$\|u_h - u\|_{1,\omega} \le Ch^{s-1}\|u\|_{s,\Omega},$$

Proof: In Lemmas 3 and 4 we have proved the estimates for the components $\eta_{i,j}$ and $\mu_{i,j}$ of the local truncation error. Hence

$$
\begin{aligned}
\|\eta\|_{*,\omega} &= \left(\sum_{x_i \in \omega} \mathrm{m}(\gamma_{ij}) d(x_i, x_j) \sum_{j \in \Sigma(i)} \eta_{i,j}^2 \right)^{1/2} \\
&\leq C \left(\sum_{x_i \in \omega} h^d \sum_{j \in \Sigma(i)} h^{2s-d-2} |u|_{s,e_{ij}}^2 \right)^{1/2} \\
&\leq C_1 h^{s-1} |u|_{s,\Omega}\,, \qquad \frac{d}{2} < s \leq 2\,.
\end{aligned}
$$

In the same way we show that

$$
\|\mu\|_{*,\omega} \leq C h^s \|\mathbf{b}\|_{d/2+\alpha,\infty,\Omega} \|u\|_{s,\Omega}
$$

when **CDS** is used, and

$$
\|\mu\|_{*,\omega} \leq C \left(h^s \|\mathbf{b}\|_{d/2+\alpha,\infty,\Omega} \|u\|_{s,\Omega} + h |\mathbf{b}|_{0,\infty,\Omega} |u|_{1,\Omega} \right)
$$

otherwise. This completes the proof. □

6 Numerical results

In this section we present numerical experiments with several model problems on a domain Ω - a pentagon with a square hole inside. The coordinates of the pentagon vertexes are $((0,0), (1,0), (1.6,0), (0.5,2), (-0.6,1))$ and the coordinates of the square hole are $((0.15, 0.45), (0.85, 0.45), (0.85, 1.15), (0.15, 1.15))$. The domain Ω is shown on Fig. 4 (with 336 mesh points). We choose such a domain to illustrate the flexibility of finite volume methods.

The Voronoi meshes are generated using the software product **triangle** and Fig. 4 is produced by **showme**. Both products are developed by J. R. Shewchuk [32]. **Triangle** is a Delauney triangulator and it provides only the control on the maximum area of the triangles. We used this option to generate six Delauney triangulations and their dual Voronoi meshes. Although this approach do not guarantee formally that the maximum area of Voronoi volumes will decrease four time on every successive level, we believe that this is true in asymptotic. The number of nodes increase roughly by four, at least in fourth, fifth and sixth triangulations. Note that the usual refinement by dividing every triangle into four equal triangles does not necessarily produce a Delauney triangulation.

The exact solution is chosen to be $u = x(1-x)y(1-y)$. The problems are:

Problem 1 (Laplacian)

$$
a(x) = 1\,, \qquad \mathbf{b}(x) = 0.
$$

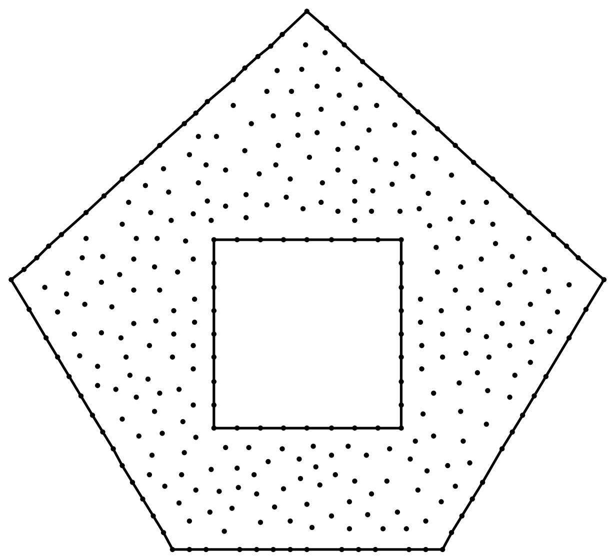

Figure 4: Domain Ω and mesh points

Problem 2 (Diffusion dominated)

$$a(x) = 1 + x^2 + y^3 \,, \qquad b_1 = (x+1)y \,, \quad b_2 = x + \sin(y).$$

Problem 3 (Convection dominated)

$$a(x) = 10^{-2} \,, \qquad b_1 = (1+x\cos(\alpha))\cos(\alpha) \,, \quad b_2 = (1+y\sin(\alpha))\sin(\alpha), \ \ \alpha = 15^{\circ}.$$

Problem 4 (Strongly convection dominated)

$$a(x) = 10^{-5} \,, \qquad b_1 = 1 \,, \quad b_2 = 0.$$

In Table 1 we report the discrete L^2 and H^1-norms of the error and the rate of convergence β, i.e. h^{β}. The results for problems 1 and 2 clearly show second order accuracy in L^2-norm and first order inH^1-norm. We observe some fluctuations that can be explained with geometrical irregularities or/and not exact decreasing of the area in consecutive triangulations mention above. (Compare with the results reported in [19]).

We point out that our theory for convection dominated case have to be considered as asymptotic, i.e., for sufficiently small h. Results reported for problems 3 and 4 show first order in L^2-norm. For $a = 10^{-2}$ the rate of convergence in H^1-norm increases with refinement of the mesh. For $a = 10^{-5}$ this stage is still not reached, although the results show that the **UDS** is stable.

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Table 1: Discrete norms of the error and rate of convergence

$N \setminus problem$		1	2	3	4
31	L^2	$4.947.10^{-3}$	$7.215.10^{-3}$	$3.166.10^{-2}$	$4.474.10^{-2}$
	β				
	H^1	$2.634.10^{-2}$	$3.843.10^{-2}$	$1.690.10^{-1}$	$2.382.10^{-1}$
	β				
97	L^2	$2.266.10^{-3}$	$2.967.10^{-3}$	$1.750.10^{-2}$	$3.432.10^{-2}$
	β	1.126	1.282	0.855	0.383
	H^1	$2.831.10^{-2}$	$3.373.10^{-2}$	$1.653.10^{-1}$	$2.602.10^{-1}$
	β	-0.104	0.188	0.032	-0.527
336	L^2	$6.474.10^{-4}$	$8.178.10^{-4}$	$1.143.10^{-1}$	$2.149.10^{-2}$
	β	1.807	1.859	0.615	0.675
	H^1	$1.265.10^{-2}$	$1.420.10^{-2}$	$1.587.10^{-1}$	$2.519.10^{-1}$
	β	1.162	1.248	0.059	0.047
1202	L^2	$1.357.10^{-4}$	$1.758.10^{-4}$	$6.205.10^{-3}$	$1.227.10^{-2}$
	β	2.254	2.212	0.881	0.808
	H^1	$7.104.10^{-3}$	$8.062.10^{-5}$	$1.195.10^{-1}$	$2.163.10^{-1}$
	β	0.832	0.817	0.409	0.220
4643	L^2	$3.359.10^{-5}$	$4.107.10^{-5}$	$3.267.10^{-3}$	$6.409.10^{-3}$
	β	2.014	2.098	0.925	0.937
	H^1	$3.511.10^{-3}$	$3.992.10^{-3}$	$9.507.10^{-2}$	$1.925.10^{-1}$
	β	1.017	1.014	0.324	0.168
18054	L^2	$9.434.10^{-6}$	$1.171.10^{-5}$	$1.664.10^{-3}$	$3.317.10^{-3}$
	β	1.832	1.810	0.973	0.950
	H^1	$1.857.10^{-3}$	$2.111.10^{-3}$	$6.494.10^{-2}$	$1.831.10^{-1}$
	β	0.919	0.919	0.544	0.072

References

[1] R. A. Adams. *Sobolev spaces.* Academic Press, New York, 1975.

[2] Todd Arbogast, Mary F. Wheeler, and Ivan Yotov. Mixed finite elements for elliptic problems with tensor coefficients as cell–centered finite differences. *SIAM J.Numer. Anal.*, to appear.

[3] Franz Aurenhammer. Voronoi diagrams – A survey of a fundamental geometric data structure. *ACM Comput. Surveys*, 23:345–405, 1991.

[4] Kinji Baba and Masahisa Tabata. On a conservative upwind finite element scheme for convection diffusion equations. *R.A.I.R.O. Analyse numerique/Numerical Analysis*, 15(1):3–25, 1981.

[5] Marshall Bern and David Eppstein. Mesh generation and optimal triangulation. In D. Z. Du and F. K. Hwang, editors, *Computing in Euclidean Geometry*, volume 1 of *Lecture Notes Series of Computing*, pages 23–90. World Scientific, 1992.

[6] J. H. Bramble and S. R. Hilbert. Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation. *SIAM J.Numer. Anal.*, 7:113–124, 1970.

[7] Z. Cai. On the finite volume element method. *Numer. Math.*, 58:713–735, 1991.

[8] Phillipe G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland Publishers, Amsterdam, 1978.

[9] Richard Ewing, Raytcho Lazarov, and Panayot Vassilevski. Local refinement techniques for elliptic problems on cell-centered grids I. Error analysis. *Math. Comp.*, 56(194):437–461, 1991.

[10] Steven Fortune. Voronoi diagrams and Delaunay triangulations. In D. Z. Du and F. K. Hwang, editors, *Computing in Euclidean Geometry*, volume 1 of *Lecture Notes Series of Computing*, pages 193–233. World Scientific, 1992.

[11] P. L. George. *Automatic mesh generation. Application to finite element methods.* John Wiley and Sons, Masson, France, 1991.

[12] Z. E. Heinemann, C.W. Brand, M. Munke, and Y.M. Chen. Modeling reservoir geometry with irregular grids. *SPE Reservoir Engineering*, pages 225–232, May 1991.

[13] B. Heinrich. *Finite Difference Methods on Irregular Networks.* Akademie-Verlag, Berlin, 1987.

[14] J. C. Heinrich, P. S. Hyakorn, O. C. Zienkiewicz, and A. R. Mitchell. An “upwind” finite element scheme for two-dimensional convective transport equations. *Int. J. Num. Meth. Engin.*, 11:131–143, 1977.

[15] R. Herbin. An error estimate for a finite volume scheme for a diffusion-convection problem on a triangular mesh. *Numer. Meth. for Partial Diff. Equat.*, 11(2):165–174, 1995.

[16] T. Kerhoven and J. W. Jerome. L_∞ stability of finite element approximations of elliptic gradient equations. *Numer. Math.*, 57:561–575, 1990.

[17] D. S. Kershaw. Differencing of the diffusion equation in lagrangian hydrodynamic codes. *J. Comput. Phys*, 39:375–395, 1987.

[18] H. O. Kreiss, T. A. Manteuffel, B. Swartz, B. Wendroff, and A. B. White, Jr. Supra–convergent schemes on irregular grids. *Math. Comp.*, 47(176):537–554, 1986.

[19] R. D. Lazarov, I. D. Mishev, and P. S. Vassilevski. Finite volume methods for convection-diffusion problems. *SIAM J.Numer. Anal.*, 33(1):31–55, 1996.

[20] J. L. Lions and E. Magenes. *Non–Homogeneous Boundary Value Problems and Applications.* Springer–Verlag, New York, 1972.

[21] Mingjin Liu, Junping Wang, and Ning Yan. A discretization for convection dominated diffusion problems based combining the mixed method with the discontinuous Galerkin procedure. 1995. University of Wyoming, in progress.

[22] J. A. Mackenzie and K. W. Morton. Finite volume solutions of convection–diffusion test problems. *Math. Comp.*, 60(201):189–220, 1992.

[23] T.A. Manteuffel and A. B. White, Jr. The numerical solution of second order boundary value problems on nonuniform meshes. *Math. Comp.*, 47(176):511–535, 1986.

[24] S. F. McCormick. *Multilevel Adaptive Methods for Partial Differential equations.* SIAM, Philadelphia, 1989.

[25] K. W. Morton and E. Süli. Finite volume methods and their analysis. *IMA J.Numer. Anal.*, 11:241–260, 1991.

[26] C. L. Palagi and Khalid Aziz. Modeling vertical and horizontal wells with voronoi grids. *SPE Reservoir Engineering*, pages 15–21, February 1994.

[27] T. Russell, J. Jones, Z. Cai, , and S. F. McCormick. A control volume mixed method on irregular quadrilateral and hexalateral grids. 1996. submitted.

[28] A.A. Samarskii. *Theory of Difference Schemes.* Nauka, Moscow, 1983. (Russian).

[29] Alexander A. Samarskii, Raytcho D. Lazarov, and Vladimir L. Makarov. *Difference Schemes for Differential Equations having Generalized Solutions.* Vysshaya Shkola Publishers, Moscow, 1987. (Russian).

[30] T. Schmidt. Box schemes on quadrilateral meshes. *Computing*, 51:271–292, 1993.

[31] Mikhail Shashkov. *Conservative Finite-difference methods on general grids.* CRC Press, New York, 1996.

[32] J. R. Shewchuk. Triangle: Engineering a 2D quality mesh generator and Delaunay triangulator. In *Proc. First Workshop on Applied Computational Geometry, Philadelphia, Pennsylvania*, pages 124–133, ACM, 1996.

[33] E. Süli. The accuracy of cell vertex finite volume methods on quadrilateral meshes. *Math. Comp.*, 59(200):359–382, 1992.

[34] J. M. Thomas and D. Trujillo. Analysis of finite volume methods. Technical Report 19, Universite de Pau et des Pays de L'adour, Pau, France, 1995.

[35] J. M. Thomas and D. Trujillo. Convergence of finite volume methods. Technical Report 20, Universite de Pau et des Pays de L'adour, Pau, France, 1995.

[36] A. N. Tikhonov and A. A. Samarskii. Homogeneous difference schemes on nonuniform nets. *Zh. Vychisl. Mat. i Mat. Fiz,*, 2:812–832, 1962. (Russian).

[37] P. S. Vassilevski, S. I. Petrova, and R. D. Lazarov. Finite difference schemes on triangular cell-centered grids with local refinement. *SIAM J. Sci. Stat. Comput.*, 13:1287–1313, 1992.

[38] A. Weiser and M. F. Wheeler. On convergence of block-centered finite differences for elliptic problems. *SIAM J. Numer. Anal.*, 25:351–375, 1988.