COMPARISON OF VARIOUS FORMULATIONS OF THREE-PHASE FLOW IN POROUS MEDIA

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ABSTRACT. Various formulations of the governing equations that describe three-phase flow in porous media, including phase, global, and pseudo-global pressure-saturation formulations, are discussed in this paper. A comparison of these differential formulations is theoretically and numerically presented for the first time. It is shown that the global pressure-saturation formulation of the governing equations offers the potential for significant improvements in the efficiency of numerical schemes.

1. INTRODUCTION

It has been shown that the governing equations describing two-phase flow in porous media can be written in a fractional flow formulation; i.e., in terms of a global pressure and saturation [1], [8], [13]. Further, it has been proven that this fractional flow approach is far more efficient than the original two-pressure approach from the computational point of view [5], [11], [12]. The main reasons for this are that the differential equations written in the fractional flow formulation formally resemble the governing equations for single-phase flow, and that efficient numerical schemes can be devised to take advantage of many physical properties inherent in the flow equations.

In this paper we discuss various formulations of the governing equations describing three-phase flow in porous media. We show that, under a so-called total differential condition on the shape of three-phase relative permeability and capillary pressure functions, the governing equations can be written in a fractional flow formulation; i.e., in terms of a global pressure and two saturations. The case of three-phase flow is quite different from the case of two-phase flow. The three-phase relative permeability and capillary pressure curves are far more complicated than the corresponding two-phase curves. It is the complexity of these three-phase curves that complicates the derivation of the global pressure-saturation form for the former case. In the two-phase flow, the governing equations can be written in terms of a global pressure and saturation without any hypothesis [13]. However, in

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the three-phase flow we show that the total differential condition is necessary and sufficient for the governing equations to be written in terms of a global pressure and two saturations. While this condition is not satisfied for all the existing three-phase curves, it is here verified that it is satisfied for some simplified models.

For the above reason on the total differential condition, we also derive other formulations of the governing equations for three-phase flow in porous media. We show that these equations can be written in terms of a phase or pseudo-global pressure and two saturations without any assumption. However, it turns out that the phase and pseudo-global pressure-saturation forms are much more complicated than the global pressure-saturation form. Especially, the coupling between the pressure and saturation equations in the phase and pseudo-global pressure-saturation forms is stronger, and thus these equations are more expensive to solve. This agrees with our theoretical and numerical observations, which are carried out here for the first time. The beauty of the global pressure-saturation form is that the three-phase differential system derived via it resembles that for the single-phase flow. Hence the differential equations in this form can be more easily solved.

In the next section we review the governing equations for three-phase flow in a porous medium. Then the phase, global, and pseudo-global pressure-saturation forms with and without a weighted total velocity are derived in §3, §4, §5, respectively. A theoretical comparison of these forms is presented in §6. The comparison of these forms via numerical experiments is given in §7; finite element methods are applied to solve the partial differential equations. Finally, a concluding remark is given in §8.

2. THE GOVERNING EQUATIONS

The usual equations describing the flow of three immiscible fluids in a porous medium $\Omega \subset \Re^3$ are given by the mass balance equation and Darcy's law for each of the fluid phases [4], [20]:

$$(2.1a) \qquad \frac{\partial(\phi \rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha u_\alpha) = q_\alpha, \qquad x \in \Omega, \, t > 0,$$

$$(2.1b) \qquad u_\alpha = -\frac{k \, k_{r\alpha}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha \tilde{g}), \qquad x \in \Omega, \, t > 0,$$

where ϕ and k are the porosity and absolute permeability of the porous medium, ρ_α, s_α, p_α, u_α, and μ_α are, respectively, the density, (reduced) saturation, pressure, volumetric velocity, and viscosity of the α-phase, q_α is the source/sink term, $k_{r\alpha}$ is the relative permeability of the α-phase, and $\tilde{g}$ is the gravitational, downward-pointing, constant vector. Below $\alpha = w$, o, and g denote water, oil, and gas phases, respectively, for example. In addition to (2.1), we also have the customary property for the saturations:

$$(2.2) \qquad \sum_\alpha s_\alpha = 1,$$

where (and later) $\sum_\alpha = \sum_{\alpha=w,o,g}$, and define, for notational convenience, the capillary pressure functions:

$$(2.3) \qquad\qquad p_{c\alpha o} = p_\alpha - p_o, \quad \alpha = w, o, g,$$

where $p_{coo} \equiv 0$, p_{cgo} represents the gas phase capillary pressure, and p_{cwo} is the negative water phase capillary pressure.

The dependent variables are s_α, p_α, and u_α. In (2.1) and (2.2), we have utilized the reduced saturations s_α, which are related to the phase saturations $\tilde{s}_\alpha$ by

$$s_\alpha = \frac{\tilde{s}_\alpha - \tilde{s}_{r\alpha}}{1 - \tilde{s}_{rw} - \tilde{s}_{ro} - \tilde{s}_{rg}}, \quad \alpha = w, o, g,$$

where $\tilde{s}_{r\alpha}$ is the residual saturation of the α-phase, $\alpha = w, o, g$. The porosity ϕ can be a function of space and pressures, and the absolute permeability k can depend on space and any dependent variables. The density ρ_α and viscosity μ_α are functions of pressures. Finally, we assume that the capillary pressure and relative permeability functions depend upon the saturations s_α solely. For notational simplicity, we neglect their dependence on space, which would then introduce lower-order terms in the partial differential equations [8], [13]. In the next three sections we shall write equation (2.1)–(2.3) in terms of a pressure p and the two saturations s_w and s_g.

3. PHASE FORMULATION

In this section the phase pressure-saturation formulation is derived.

3.1. Phase pressure-saturation formulation. For expositional convenience, we introduce the phase mobility functions

$$\lambda_\alpha = k_{r\alpha}/\mu_\alpha, \quad \alpha = w, o, g,$$

and the total mobility

$$\lambda = \sum_\alpha \lambda_\alpha.$$

Also, we define the fractional flow functions

$$f_\alpha = \lambda_\alpha/\lambda, \quad \alpha = w, o, g.$$

We see that $\sum_\alpha f_\alpha = 1$.

We use the oil phase pressure as the pressure variable in this section:

$$(3.1) \qquad\qquad p = p_o,$$

and define the total velocity

$$(3.2) \qquad u = \sum_\alpha u_\alpha.$$

Then, use (3.1) and (3.2), carry out the differentiation indicated in (2.1a), divide by ρ_α in (2.1a), and apply (2.2) and (2.3) to obtain the differential equations with $x \in \Omega$ and $t > 0$:

$$(3.3a) \qquad u = -k\lambda(\nabla p - G_\lambda + \sum_\alpha f_\alpha \nabla p_{c\alpha o}),$$

$$(3.3b) \qquad \nabla \cdot u = -\frac{\partial \phi}{\partial t} + \sum_\alpha \frac{1}{\rho_\alpha}\left(q_\alpha - \phi s_\alpha \frac{\partial \rho_\alpha}{\partial t} - u_\alpha \cdot \nabla \rho_\alpha\right),$$

and

$$(3.4) \qquad \begin{aligned} &\phi\frac{\partial s_\alpha}{\partial t} + \nabla \cdot \left\{f_\alpha u + kf_\alpha \sum_\beta \lambda_\beta\left(\nabla(p_{c\beta o} - p_{c\alpha o}) - (\rho_\beta - \rho_\alpha)\tilde{g}\right)\right\} \\ &= -s_\alpha\frac{\partial \phi}{\partial t} + \frac{1}{\rho_\alpha}\left(q_\alpha - \phi s_\alpha\frac{\partial \rho_\alpha}{\partial t} - u_\alpha \cdot \nabla \rho_\alpha\right), \quad \alpha = w, g, \end{aligned}$$

where

$$G_\lambda = \tilde{g}\sum_\alpha f_\alpha \rho_\alpha.$$

The equations in (3.3) and (3.4) are, respectively, the pressure and saturation equations. The phase velocity is related to the total velocity by

$$(3.5) \qquad u_\alpha = f_\alpha u + kf_\alpha \sum_\beta \lambda_\beta\left(\nabla(p_{c\beta o} - p_{c\alpha o}) - (\rho_\beta - \rho_\alpha)\tilde{g}\right), \quad \alpha = w, o, g.$$

3.2. Phase formulation with a weighted total velocity. In the right-hand sides of (3.3b) and (3.4) appear the terms $u_\alpha \cdot \nabla \rho_\alpha$, which are essentially quadratic in the velocities. To get rid of these terms, we now introduce a weighted total velocity. Toward that end, set

$$\lambda_\alpha = k_{r\alpha}\rho_\alpha/\mu_\alpha, \ \lambda = \sum_\beta \lambda_\beta, \quad \alpha = w, o, g,$$

and

$$f_\alpha = \lambda_\alpha/\lambda, \quad \alpha = w, o, g.$$

The pressure variable is defined as in (3.1), but the total velocity is now weighted:

$$(3.6) \qquad u = \sum_\alpha \rho_\alpha u_\alpha.$$

Then with the same manipulation on (2.1) as above, we have the pressure and saturation equations with $x \in \Omega$ and $t > 0$:

$$(3.7a) \qquad u = -k\lambda\Big(\nabla p - G_\lambda + \sum_\alpha f_\alpha \nabla p_{c\alpha o}\Big),$$

$$(3.7b) \qquad \sum_\alpha \frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot u = \sum_\alpha q_\alpha,$$

and

$$(3.8) \qquad \frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot \Big\{ f_\alpha u + k f_\alpha \sum_\beta \lambda_\beta \big(\nabla(p_{c\beta o} - p_{c\alpha o}) - (\rho_\beta - \rho_\alpha)\tilde{g} \big) \Big\}$$
$$= q_\alpha, \quad \alpha = w, g.$$

The phase velocity is given by

$$u_\alpha = \rho_\alpha^{-1}\Big\{ f_\alpha u + k f_a \sum_\beta \lambda_\beta \big(\nabla(p_{c\beta o} - p_{c\alpha o}) - (\rho_\beta - \rho_\alpha)\tilde{g} \big) \Big\}, \quad \alpha = w, o, g.$$

4. GLOBAL FORMULATION

In this section the global pressure-saturation formulation is derived.

4.1. Global pressure-saturation formulation. The phase and total mobilities and the fractional flow functions are defined in the same manner as in §3.1; i.e.,

$$\lambda_\alpha = k_{r\alpha}/\mu_\alpha, \ \lambda = \sum_\beta \lambda_\beta, \ f_\alpha = \lambda_\alpha/\lambda, \quad \alpha = w, o, g.$$

To introduce a global pressure, we assume that the fractional flow functions f_α depend solely on the saturations s_w and s_g (for pressure-dependent functions f_α, see the next subsection), and that there exists a function $(s_w, s_g) \longmapsto p_c(s_w, s_g)$ such that

$$(4.1) \qquad \nabla p_c = f_w \nabla p_{cwo} + f_g \nabla p_{cgo}.$$

This holds if and only if the following equations are satisfied:

$$(4.2a) \qquad \frac{\partial p_c}{\partial s_w} = f_w \frac{\partial p_{cwo}}{\partial s_w} + f_g \frac{\partial p_{cgo}}{\partial s_w},$$

$$(4.2b) \qquad \frac{\partial p_c}{\partial s_g} = f_w \frac{\partial p_{cwo}}{\partial s_g} + f_g \frac{\partial p_{cgo}}{\partial s_g}.$$

A necessary and sufficient condition for existence of a function p_c satisfying (4.2) is

$$(4.3) \qquad \frac{\partial f_w}{\partial s_g}\frac{\partial p_{cwo}}{\partial s_w} + \frac{\partial f_g}{\partial s_g}\frac{\partial p_{cgo}}{\partial s_w} = \frac{\partial f_w}{\partial s_w}\frac{\partial p_{cwo}}{\partial s_g} + \frac{\partial f_g}{\partial s_w}\frac{\partial p_{cgo}}{\partial s_g}.$$

This condition is referred to as the total differential condition [8]. When the condition (4.3) is satisfied, the function p_c is determined by

$$
\begin{aligned}
(4.4) \qquad p_c(s_w, s_g) = & \int_1^{s_w}\left\{f_w(\xi, 0)\frac{\partial p_{cwo}}{\partial s_w}(\xi, 0) + f_g(\xi, 0)\frac{\partial p_{cgo}}{\partial s_w}(\xi, 0)\right\} d\xi \\
& + \int_0^{s_g}\left\{f_w(s_w, \xi)\frac{\partial p_{cwo}}{\partial s_g}(s_w, \xi) + f_g(s_w, \xi)\frac{\partial p_{cgo}}{\partial s_g}(s_w, \xi)\right\} d\xi,
\end{aligned}
$$

where we assume that the above integrals are well-defined, which is always true in practical situations [8]. We now introduce the global pressure by

$$(4.5) \qquad p = p_o + p_c,$$

and the total velocity by

$$(4.6) \qquad u = \sum_\alpha u_\alpha.$$

Now, use the condition (4.3), the definitions in (4.4)–(4.6), and the same calculations as in §3.1 to get the pressure and saturation equations with $x \in \Omega$ and $t > 0$:

$$(4.7a) \qquad u = -k\lambda(\nabla p - G_\lambda),$$

$$(4.7b) \qquad \nabla \cdot u = -\frac{\partial \phi}{\partial t} + \sum_\alpha \frac{1}{\rho_\alpha}\left(q_\alpha - \phi s_\alpha \frac{\partial \rho_\alpha}{\partial t} - u_\alpha \cdot \nabla \rho_\alpha\right),$$

and

$$
\begin{aligned}
(4.8) \qquad & \phi\frac{\partial s_\alpha}{\partial t} + \nabla \cdot \left\{f_\alpha u + k\lambda_\alpha\left(\nabla(p_c - p_{c\alpha o}) - \delta_\alpha\right)\right\} \\
& = -s_\alpha\frac{\partial \phi}{\partial t} + \frac{1}{\rho_\alpha}\left(q_\alpha - \phi s_\alpha\frac{\partial \rho_\alpha}{\partial t} - u_\alpha \cdot \nabla\rho_\alpha\right), \quad \alpha = w, g,
\end{aligned}
$$

where

$$\delta_\alpha = (f_\beta(\rho_\beta - \rho_\alpha) + f_\gamma(\rho_\gamma - \rho_\alpha))\,\tilde{g}, \quad \alpha, \beta, \gamma = w, o, g,\ \alpha \neq \beta,\ \beta \neq \gamma,\ \gamma \neq \alpha.$$

Finally, the phase velocity is determined by

$$(4.9) \qquad u_\alpha = f_\alpha u + k\lambda_\alpha\left(\nabla(p_c - p_{c\alpha o}) - \delta_\alpha\right), \quad \alpha = w, o, g.$$

4.2. Global formulation with a weighted total velocity. As in §3.2, to get rid of the quadratic terms in the velocities in (4.7b), we define

$$\lambda_\alpha = k_{r\alpha}\rho_\alpha/\mu_\alpha, \ \lambda = \sum_\beta \lambda_\beta, \quad \alpha = w, o, g,$$

and

$$f_\alpha = \lambda_\alpha/\lambda, \quad \alpha = w, o, g.$$

Also, define the weighted total velocity

$$(4.10) \qquad\qquad u = \sum_\alpha \rho_\alpha u_\alpha.$$

In the present case we assume that the fractional flow functions f_α depend on the saturations s_w and s_g and a pressure p, and that there exists a function $(s_w, s_g, p) \longmapsto p_c(s_w, s_g, p)$ satisfying

$$(4.11) \qquad\qquad \nabla p_c = f_w \nabla p_{cwo} + f_g \nabla p_{cgo} + \frac{\partial p_c}{\partial p} \nabla p.$$

The assumption on the dependence on the pressure p means that we ignore the error caused by calculating the density and viscosity functions for the α-phase at p instead of p_α. For details on this error, the reader is referred to [13] for a similar treatment as in the two-phase flow.

With the same argument as in §4.1, a necessary and sufficient condition for existence of a function p_c satisfying (4.11) is (4.3); i.e.,

$$(4.12) \qquad \frac{\partial f_w}{\partial s_g}\frac{\partial p_{cwo}}{\partial s_w} + \frac{\partial f_g}{\partial s_g}\frac{\partial p_{cgo}}{\partial s_w} = \frac{\partial f_w}{\partial s_w}\frac{\partial p_{cwo}}{\partial s_g} + \frac{\partial f_g}{\partial s_w}\frac{\partial p_{cgo}}{\partial s_g},$$

where p is treated as a parameter. Under the condition (4.12), the function p_c is described by

$$
\begin{aligned}
(4.13) \quad p_c(s_w, s_g, p) = &\int_1^{s_w} \left\{ f_w(\xi, 0, p)\frac{\partial p_{cwo}}{\partial s_w}(\xi, 0) + f_g(\xi, 0, p)\frac{\partial p_{cgo}}{\partial s_w}(\xi, 0) \right\} d\xi \\
&+ \int_0^{s_g} \left\{ f_w(s_w, \xi, p)\frac{\partial p_{cwo}}{\partial s_g}(s_w, \xi) + f_g(s_w, \xi, p)\frac{\partial p_{cgo}}{\partial s_g}(s_w, \xi) \right\} d\xi.
\end{aligned}
$$

The global pressure is again defined by

$$(4.14) \qquad\qquad p = p_o + p_c.$$

Then, as before, we have the pressure and saturation equations with $x \in \Omega$ and $t > 0$:

$$(4.15\text{a}) \qquad u = -k\lambda(\omega \nabla p - G_\lambda),$$

$$(4.15\text{b}) \qquad \sum_\alpha \frac{\partial(\phi \rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot u = \sum_\alpha q_\alpha,$$

and

$$(4.16) \qquad \frac{\partial(\phi \rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot \left\{ \omega^{-1} f_\alpha u + k\lambda_\alpha \left(\nabla(p_c - p_{c\alpha o}) - \delta_\alpha \right) - \omega^{-1} \frac{\partial p_c}{\partial p} G_\lambda \right\}$$
$$= q_\alpha, \quad \alpha = w, g,$$

where

$$\omega(s_w, s_g, p) = 1 - \frac{\partial p_c}{\partial p}.$$

The phase velocity is computed by

$$(4.17) \quad u_\alpha = \rho_\alpha^{-1} \left\{ \omega^{-1} f_\alpha u + k\lambda_\alpha \left(\nabla(p_c - p_{c\alpha o}) - \delta_\alpha \right) - \omega^{-1} \frac{\partial p_c}{\partial p} G_\lambda \right\}, \quad \alpha = w, o, g.$$

4.3. Total differential condition. In this subsection we discuss the total differential condition (4.3). For three-phase flow, the classical capillary pressures [18] are normally used:

$$(4.18) \qquad p_{cwo} = p_{cwo}(s_w), \quad p_{cgo} = p_{cgo}(s_g).$$

Typical normalized capillary pressure functions are shown in Figures 1 and 2.
   Using (4.18), the condition (4.3) reduces to

$$(4.19) \qquad \frac{\partial f_w}{\partial s_g} \frac{\partial p_{cwo}}{\partial s_w} = \frac{\partial f_g}{\partial s_w} \frac{\partial p_{cgo}}{\partial s_g}.$$

Also, we have the usual definitions of the relative permeabilities

$$k_{rw} = k_{rw}(s_w), \ k_{ro} = k_{ro}(s_w, s_g), \ k_{rg} = k_{rg}(s_g).$$

Typical relative permeability curves are given in Figures 3 and 4. Then equation (4.19) can be simplied further:

$$(4.20) \qquad \lambda_w \frac{\partial \lambda}{\partial s_g} \frac{\partial p_{cwo}}{\partial s_w} = \lambda_g \frac{\partial \lambda}{\partial s_w} \frac{\partial p_{cgo}}{\partial s_g}.$$
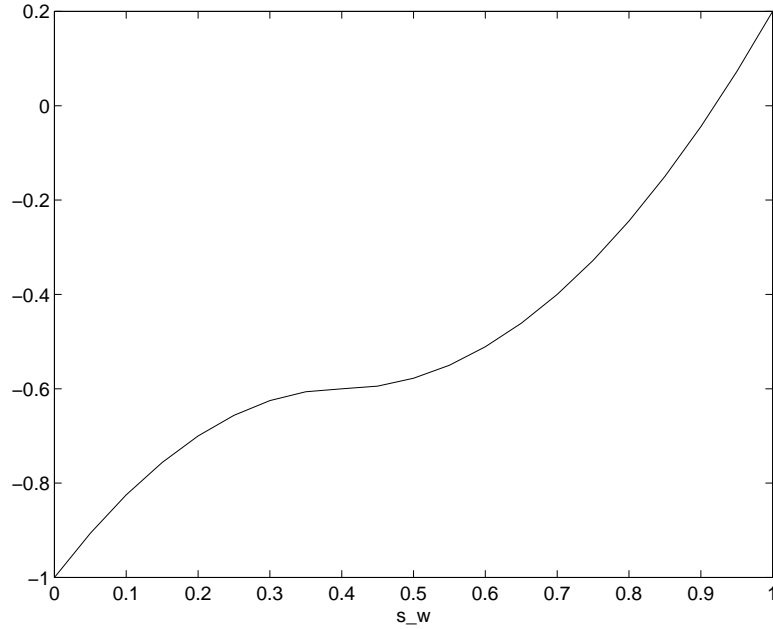
FIG. 1. The typical normalized capillary pressure p_{cwo}.

We can construct three-phase relative permeability and capillary pressure curves which satisfy the condition (4.20). A simple numerical procedure for constructing these curves has been described in [8]. Some of the numerical examples satisfying (4.20) have been compared with the classical Stone's model [22], which does not satisfy this condition, and similar results were obtained. Here we will see that some simplified three-phase models in fact (or approximately) satisfy the condition (4.20).

It follows from (4.20) that, if the total mobility λ is close to a constant function, then the total differential condition holds approximately. We now consider the following cases. The simplied Corey's and Baker's models [14], [15] for the three-phase relative permeabilities have the form

$$k_{rw} = s_w^{e_w}, \ k_{ro} = (1 - s_w - s_g)^{e_o}, \ k_{rg} = s_g^{e_g},$$

where the e_α's are constants. Substitute them into (4.20) to see that

(4.21)
$$\frac{s_w^{e_w}}{\mu_w}\left\{\frac{e_g}{\mu_g}s_g^{e_g-1} - \frac{e_o}{\mu_o}(1 - s_w - s_g)^{e_o-1}\right\}\frac{\partial p_{cwo}}{\partial s_w}$$
$$= \frac{s_g^{e_g}}{\mu_g}\left\{\frac{e_w}{\mu_w}s_w^{e_w-1} - \frac{e_o}{\mu_o}(1 - s_w - s_g)^{e_o-1}\right\}\frac{\partial p_{cgo}}{\partial s_g}.$$
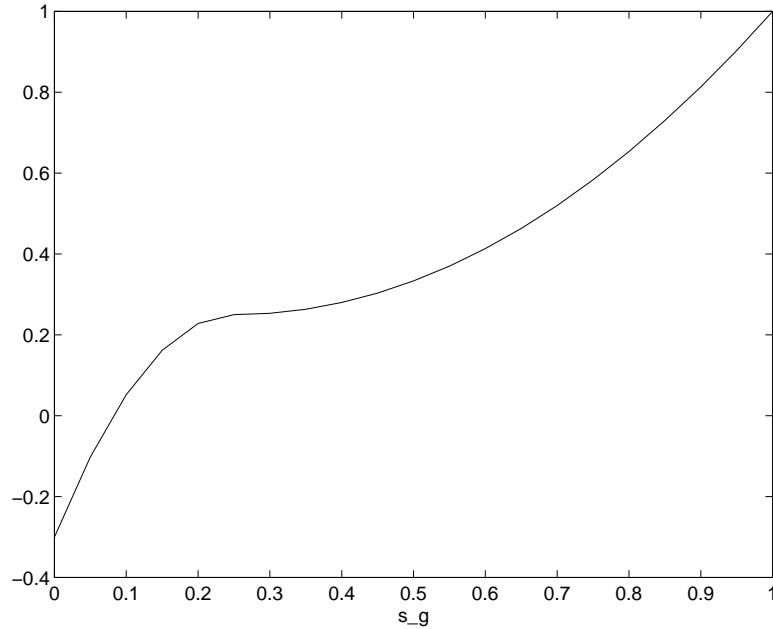
FIG. 2. The typical normalized capillary pressure p_{cgo}.

Now, we see that the total differential condition is identically satisfied in the case of compressible fluids with cross-relative permeabilities where $e_\alpha = 1$, $\alpha = w, o, g$, and unity viscosities. For other choices of e_α, equation (4.21) depends on the definitions of the capillary pressures p_{cwo} and p_{cgo}, and can be approximately (if not identically) satisfied by appropriate choices of parameters.

5. PSEUDO-GLOBAL FORMULATION

The global formulation in §4 requires the total differential condition (4.3) on the shape of three-phase relative permeability and capillary pressure functions. In this section we derive a pseudo-global pressure-saturation formulation, which does not require this condition.

5.1. Pseudo-global pressure-saturation formulation. Again, the phase and total mobilities and the fractional flow functions are defined as in §3.1:

$$\lambda_\alpha = k_{r\alpha}/\mu_\alpha, \ \lambda = \sum_\beta \lambda_\beta, \ f_\alpha = \lambda_\alpha/\lambda, \quad \alpha = w, o, g.$$

To introduce a pseudo-global pressure, we assume that the fractional flow functions f_α depend solely on the saturations s_w and s_g (for pressure-dependent functions f_α,
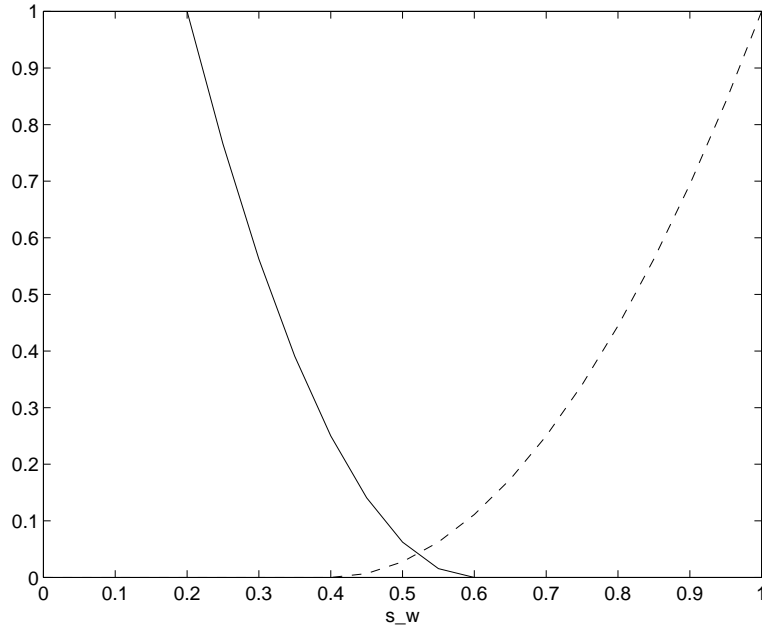
FIG. 3. Typical water-oil imbibition relative
permeabilities k_{ro} (left) and k_{rw} (right).

see the next subsection). Also, assume that the capillary pressures satisfy (4.18).
Then it follows from (3.3a) that

$$(5.1) \qquad u = -k\lambda\Big(\nabla p_o - G_\lambda + \sum_\alpha f_\alpha \frac{dp_{c\alpha o}}{ds_\alpha}\nabla s_\alpha\Big).$$

We introduce the mean values

$$\overline{f}_w(s_w) = \frac{1}{1 - s_w}\int_0^{1-s_w} f_w(s_w, \xi)d\xi,$$

$$\overline{f}_g(s_g) = \frac{1}{1 - s_g}\int_0^{1-s_g} f_g(\xi, s_g)d\xi,$$

and the pseudo-global pressure

$$p = p_o + \int_{s_{wc}}^{s_w}\overline{f}_w(\xi)\frac{dp_{cwo}(\xi)}{ds_w}d\xi + \int_{s_{gc}}^{s_g}\overline{f}_g(\xi)\frac{dp_{cgo}(\xi)}{ds_g}d\xi,$$

where s_{wc} and s_{gc} are such that $p_{cwo}(s_{wc}) = 0$ and $p_{cgo}(s_{gc}) = 0$. Now, by (5.1),
we see that

$$(5.2) \qquad u = -k\lambda\Big(\nabla p - G_\lambda + \sum_\alpha (f_\alpha - \overline{f}_\alpha)\frac{dp_{c\alpha o}}{ds_\alpha}\nabla s_\alpha\Big),$$
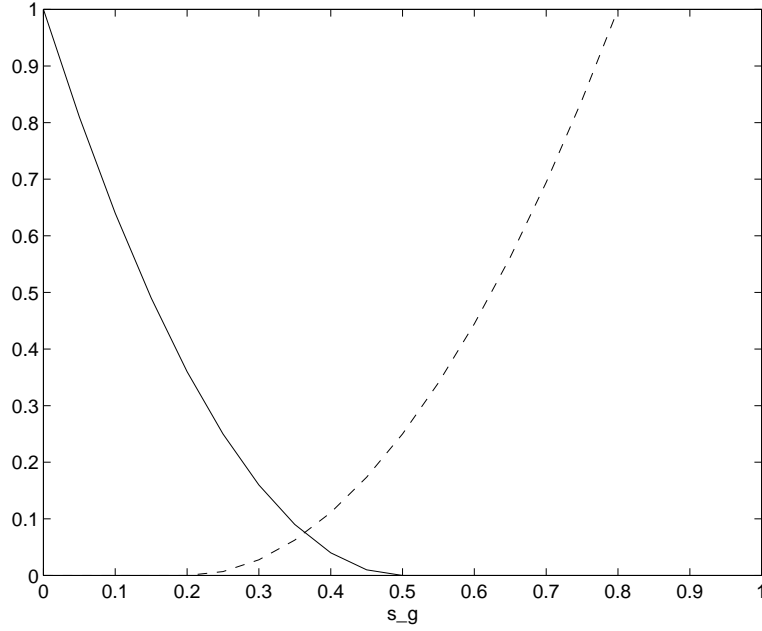
FIG. 4. Typical gas-oil drainage relative
permeabilities k_{ro} (left) and k_{rg} (right).

where $\overline{f}_o = 0$. The equations (3.3b) and (3.4) remain the same here.

5.2. Pseudo-global formulation with a weighted total velocity. The phase
and total mobilities and the fractional flow functions are again given as in §3.2:

$$\lambda_\alpha = k_{r\alpha}\rho_\alpha/\mu_\alpha, \ \lambda = \sum_\beta \lambda_\beta, \ f_\alpha = \lambda_\alpha/\lambda, \quad \alpha = w, o, g.$$

We now assume that the fractional flow functions f_α depend on the saturations s_w
and s_g and a pressure p. Then the mean values are accordingly modified by

$$\overline{f}_w(s_w, p) = \frac{1}{1 - s_w} \int_0^{1-s_w} f_w(s_w, \xi, p)d\xi,$$

$$\overline{f}_g(s_g, p) = \frac{1}{1 - s_g} \int_0^{1-s_g} f_g(\xi, s_g, p)d\xi,$$

and the pseudo-global pressure has the corresponding expression

$$p = p_o + \int_{s_{wc}}^{s_w} \overline{f}_w(\xi, p)\frac{dp_{cwo}(\xi)}{ds_w}d\xi + \int_{s_{gc}}^{s_g} \overline{f}_g(\xi, p)\frac{dp_{cgo}(\xi)}{ds_g}d\xi.$$

Apply this definition to (3.7a) to obtain

$$(5.3) \qquad u = -k\lambda\Big(\omega\nabla p - G_\lambda + \sum_\alpha (f_\alpha - \overline{f}_\alpha)\frac{dp_{c\alpha o}}{ds_\alpha}\nabla s_\alpha\Big),$$

where

$$\omega = 1 - \int_{s_{wc}}^{s_w} \frac{d}{dp}\overline{f}_w(\xi,p)\frac{dp_{cwo}(\xi)}{ds_w}d\xi - \int_{s_{gc}}^{s_g} \frac{d}{dp}\overline{f}_g(\xi,p)\frac{dp_{cgo}(\xi)}{ds_g}d\xi.$$

The other two equations (3.7b) and (3.8) remains unchanged.

6. THEORETICAL COMPARISON

We first note that if f_w and f_g are close to their respective mean values $\overline{f}_w$ and $\overline{f}_g$, then the last term in the right-hand side of (5.2) and (5.3) can be neglected. In particular, in the case of compressible fluids with cross-relative permeabilities and unity viscosities mentioned in §4.3, $f_w = \overline{f}_w$ and $f_g = \overline{f}_g$. Hence, in these cases the pseudo-global form reduces to the global form. However, in the general case the pseudo-global form is essentially the same as the phase form, and the coupling between the pressure and saturation equations in these two forms has the same pattern. Hence we only compare the phase and global forms here and in the next section; an analogous comparison between the pseudo-global and global forms can be carried out.

We compare the equations (3.3) and (3.4) with the equations (4.7) and (4.8); the same comparison between the equations (3.7) and (4.15), and the equations (3.8) and (4.16) can be done in the same way. Note that the 'continuity' equations (3.3b) and (4.7b) have the same form. However, the coupling between the pressure and saturation equations in (3.3) and (3.4) is stronger than that between the equations in (4.7) and (4.8). In particular, equation (3.3a) has the gradient of the two capillary pressure functions p_{cwo} and p_{cgo}, with different coefficients, but equation (4.7a) in form resembles the Darcy law for the single-phase flow, and is much simpler. Hence the computation of the pressure equation (3.3a) by the mixed finite element methods described in the next section requires the approximation of the two terms:

$$f_w\nabla p_{cwo} \text{ and } f_g\nabla p_{cgo}.$$

What is more, while the capillary diffusion terms involve the gradient of the two capillary pressure functions p_{cwo} and p_{cgo} in (3.4) for $\alpha = w$ and g, the calculation of the diffusion terms requires the resolution of the four linear systems

$$(\lambda_o + \lambda_g)\lambda_w\nabla p_{cwo}, \ \lambda_g\lambda_w\nabla p_{cgo}, \ \lambda_w\lambda_g\nabla p_{cwo}, \text{ and } (\lambda_o + \lambda_w)\lambda_g\nabla p_{cgo}.$$

However, in (4.8) for $\alpha = w$ and g the diffusion terms only require the resolution of the two linear systems

$$\lambda_w \nabla(p_c - p_{cwo}) \text{ and } \lambda_g \nabla(p_c - p_{cgo}).$$

Therefore, we would expect that equations (3.3) and (3.4) are more expensive from the computational point of view. This is the case, as shown in the next section.

We close this section with two remarks. First, in the uninteresting case in which $p_{cwo} = p_{cgo} \equiv 0$, the models presented in §3.1, §4.1, and §5.1 (respectively, §3.2, §4.2, and §5.2) are the same. Second, boundary conditions imposed for the three-phase flow equations can be incorporated into the fractional flow formulation in the same manner as for the two-phase flow [13].

7. NUMERICAL COMPARISON

In order to compare the efficiency of numerical schemes for solving the two systems in (3.3) and (3.4) (respectively, (3.7) and (3.8)), and in (4.7) and (4.8) (respectively, (4.15) and (4.16)), we restrict ourselves to two relatively simple sets of data. Of course, more physically adequate data could be used; however, here we are mainly interested in the comparison of these two systems.

7.1. The first test. The capillary pressure functions are defined as follows:

$$p_{cwo} = s_w - 1, \quad p_{cgo} = 1 - s_g.$$

Recall that p_{cwo} is the negative water phase capillary pressure. The relative permeability curves are given by

$$k_{rw} = s_w, \quad k_{ro} = 1 - s_w - s_g, \quad k_{rg} = s_g.$$

With these choices, the total differential condition (4.3) is satisfied from the discussion in §4.3. Further, with $\phi = \mu_\alpha = \rho_\alpha = 1$, $\alpha = w, o, g$, and $\tilde{g} = 0$, the mobility and fractional flow functions become

$$\lambda_w = f_w = s_w, \quad \lambda_o = f_o = 1 - s_w - s_g, \quad \lambda_g = f_g = s_g, \quad \lambda = 1.$$

Thus the function p_c is given by

$$p_c = -\frac{1}{2}(1 - s_w^2 + s_g^2).$$

For the present set of data, the system in (3.3) and (3.4) (respectively, (4.7) and (4.8)) is the same as that in (3.7) and (3.8) (respectively, (4.15) and (4.16)). Finally, the domain Ω is the unit cube $\Omega = (0,1)^3$, and a no-flow boundary condition for each phase is taken:

(7.1) $$u_\alpha \cdot \nu = 0, \quad \alpha = w, o, g, \ x \in \partial\Omega, \ t > 0,$$

where ν is the outer unit normal to the boundary $\partial\Omega$ of Ω.

In the phase pressure-saturation form, the equations (3.3) and (3.4) now reduce to (with $x \in \Omega$ and $t > 0$):

$$(7.2a) \qquad\qquad u = -k(\nabla p + s_w \nabla s_w - s_g \nabla s_g),$$

$$(7.2b) \qquad\qquad \nabla \cdot u = q,$$

and

$$(7.3a) \qquad \frac{\partial s_w}{\partial t} + \nabla \cdot \left\{ s_w u - k s_w \big((1 - s_w) \nabla s_w + s_g \nabla s_g \big) \right\} = q_w,$$

$$(7.3b) \qquad \frac{\partial s_g}{\partial t} + \nabla \cdot \left\{ s_g u + k s_g \big((1 - s_g) \nabla s_g + s_w \nabla s_w \big) \right\} = q_g,$$

where $q = \sum_\alpha q_\alpha$. Similarly, in the global pressure-saturation form the pressure equation (4.7) becomes

$$(7.4a) \qquad\qquad u = -k \nabla p,$$

$$(7.4b) \qquad\qquad \nabla \cdot u = q;$$

the saturation equations are the same as in (7.3) for the present set of data. Recall that the p in (7.2a) is the oil phase pressure, while the p in (7.4a) is the global pressure defined in (4.5). Also, it follows from (7.1) that the boundary condition for (7.2) and (7.4) is

$$(7.5) \qquad\qquad u \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0,$$

and that the boundary conditions for (7.3a) and (7.3b) are

$$(7.6a) \qquad \left\{ k s_w \big((1 - s_w) \nabla s_w + s_g \nabla s_g \big) \right\} \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0,$$

$$(7.6b) \qquad \left\{ k s_g \big((1 - s_g) \nabla s_g + s_w \nabla s_w \big) \right\} \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0.$$

For the present simple problem, (7.4) implies that the pressure equation is completely decoupled from the saturation equations in the global pressure-saturation form, and thus it can be independently computed and the resulting total velocity can be used by the saturation equations later. In the phase pressure-saturation form, the system in equations (7.2) and (7.3) is solved sequentially. An approximation of u is first obtained at time level $t = t^n$ from solution of equation (7.2) with the saturations s_w and s_g evaluated at the previous time level $t = t^{n-1}$. Then, using the current approximation for u, approximations of s_w and s_g are obtained at $t = t^n$ by using (7.3a) and (7.3b) simultaneously. The saturation equations are solved here by the classical up-weighting finite difference scheme, while the pressure equation is solved by a mixed finite element method. We assume that the

reader is familiar with the former scheme; the latter method will be reviewed in the appendix.

Uniform partitions of Ω into rectangular parallelepipeds with the space step $h = \Delta x = \Delta y = \Delta z$ are taken. The time differentiation terms in (7.3) are discretized with the backward Euler scheme, and the time step is assumed to be proportional to the space step: $\Delta t = \kappa h$, where κ is the proportionality constant. A cell-centered finite difference method with the seven point stencil is used for the solution of the saturation equations, while a mixed finite element method with the use of the Raviart-Thomas-Nedelec mixed space [21], [19] of lowest-order over rectangular parallelepipeds is applied to the solution of the pressure equations (see the appendix). Tables 1–4 describe the errors and convergence orders in the L^∞-norm for the pressure and saturation at $t = 1$ for the phase and global pressure-saturation differential systems, where s_h is the approximation to the water saturation. In Table 5, the CPU times in seconds for solving the whole pressure-saturation system over the given mesh up to time $t = 1$ from the initial time $t = 0$ are presented. CPU-P denotes the CPU times for the phase system, while CPU-B indicates those for the global system. All experiments are carried out on a Sun workstation.

$1/h$	L^∞-error	L^∞-order
10	0.10356	–
20	0.05147	1.01
40	0.02533	1.02

Table 1. Convergence of p_h for the phase system in test one.

$1/h$	L^∞-error	L^∞-order
10	0.23302	–
20	0.11875	0.97
40	0.05950	0.99

Table 2. Convergence of s_h for the phase system in test one.

$1/h$	L^∞-error	L^∞-order
10	0.10402	–
20	0.05208	1.00
40	0.02576	1.02

Table 3. Convergence of p_h for the global system in test one.

$1/h$	L^∞-error	L^∞-order
10	0.19847	–
20	0.09982	0.99
40	0.04977	1.00

Table 4. Convergence of s_h for the global system in test one.

$1/h$	CPU-B	CPU-P
10	30.05	56.02
20	241.21	443.14
40	965.03	1772.00

Table 5. CPU times up to $t = 1$ in test one.

It follows from Tables 1–4 that the numerical results agree with the theoretical error prediction $O(\Delta t + h)$ for both systems. However, the CPU times required for the solution of the phase pressure-saturation system almost double those for the global system. This shows that the latter system can be more easily solved, and

agrees with our theoretical observation in §6. Lots of time is spent on the coupling between the pressure and saturation equations in the former system.

7.2. The second test. In the second test we take the same set of data as in the first test except that the gas density is chosen as follows:

$$(7.7) \qquad\qquad \rho_g = 1 + p_g.$$

The comparison is here done for the two systems in (3.3) and (3.4) and in (4.7) and (4.8). Similar results have been observed for the comparison between the two systems in (3.7) and (3.8) and in (4.15) and (4.16). Set

$$(7.8) \qquad \tilde{q}_g(s_g, p) = \frac{1}{\rho_g}\left(q_g - \phi s_g \frac{\partial \rho_g}{\partial t} - u_g \cdot \nabla \rho_g\right),$$

where u_g is given by (3.5) (respectively, (4.9)). With (7.7) and (7.8), the differential equations in (3.3) and (3.4) become

$$(7.9\text{a}) \qquad\qquad u = -k(\nabla p + s_w \nabla s_w - s_g \nabla s_g),$$
$$(7.9\text{b}) \qquad\qquad \nabla \cdot u = q_w + q_o + \tilde{q}_g(s_g, p),$$

and

$$(7.10\text{a}) \qquad \frac{\partial s_w}{\partial t} + \nabla \cdot \left\{s_w u - k s_w \big((1 - s_w)\nabla s_w + s_g \nabla s_g\big)\right\} = q_w,$$
$$(7.10\text{b}) \qquad \frac{\partial s_g}{\partial t} + \nabla \cdot \left\{s_g u + k s_g \big((1 - s_g)\nabla s_g + s_w \nabla s_w\big)\right\} = \tilde{q}_g(s_g, p).$$

The equations in (4.7) reduce to

$$(7.11\text{a}) \qquad\qquad u = -k\nabla p,$$
$$(7.11\text{b}) \qquad\qquad \nabla \cdot u = q_w + q_o + \tilde{q}_g(s_g, p);$$

the saturation equations are given in the same way as in (7.10) as observed above. Finally, the boundary conditions are the same as in (7.5) and (7.6).

Note that, in the present situation, it follows from (7.7) and (7.8) that the pressure equations (7.9) and (7.11) are parabolic. Also, the pressure equation (7.11) is not totally decoupled from its saturation equations. These are the major differences between the two tests.

The above discretization techniques and set of numerical data in the first test are used here. The corresponding convergence results and CPU times are displayed in Tables 6–10. The convergence results have the same performance as in the first test. However, the difference between the CPU times for the two forms is slightly different in these two tests. The reason is that the pressure equation (7.4)

is decoupled from its saturation equations in the first test, but not in the second test. Also, because of the stronger coupling between the pressure and saturation equations in the phase form, the global system takes less time.

$1/h$	L^∞-error	L^∞-order
10	0.20207	–
20	0.10367	0.96
40	0.05160	1.01

Table 6. Convergence of p_h for the phase system in test two.

$1/h$	L^∞-error	L^∞-order
10	0.50273	–
20	0.26600	0.92
40	0.14091	0.92

Table 7. Convergence of s_h for the phase system in test two.

$1/h$	L^∞-error	L^∞-order
10	0.03991	–
20	0.01980	1.01
40	0.00983	1.01

Table 8. Convergence of p_h for the global system in test two.

We end with two remarks. First, the transport term in the saturation equations is governed by the (or weighted) total velocity u. Thus accurate numerical simulations

require an accurate approximation for u. The mixed finite element method is here used to approximate u and p simultaneously, via the coupled system of first-order differential equations in (7.2) or (7.4), and produces an accurate velocity [17]. Second, due to their convection-dominated feature, more efficient approximate procedures should be used to solve the saturation equations. However, the interest here is in the comparison between the two differential systems; the simple finite difference scheme is accurate enough for this purpose.

$1/h$	L^∞-error	L^∞-order
10	0.09843	–
20	0.05180	0.93
40	0.02701	0.94

Table 9. Convergence of s_h for the global system in test two.

$1/h$	CPU-B	CPU-P
10	51.02	73.13
20	403.00	581.00
40	1602.0	2322.03

Table 10. CPU times up to $t = 1$ in test two.

8. CONCLUDING REMARK

The phase, pseudo-global, and global pressure-saturation differential systems have been established for the three-phase fluid flow in porous media. A comparison between these systems has been carried out both theoretically and numerically. The global differential system is far more efficient than the phase and pseudo-global systems from the computational point of view, and also more suitable for mathematical analysis. The advantage of the global form can be more obviously seen in the case of incompressible flow and one-space dimension. For, in this case, the global pressure equation can be analytically solved. The weakness of the global

formulation is the need of the satisfaction of the total differential condition by the three-phase relative permeability and capillary pressure curves.

Appendix. REMARKS ON A MIXED METHOD

We rewrite equations (7.4) and (7.5) as follows:

(A.1a) $$u = -k\nabla p, \qquad x \in \Omega,$$

(A.1b) $$\nabla \cdot u = q, \qquad x \in \Omega,$$

(A.1c) $$u \cdot \nu = 0, \qquad x \in \partial\Omega.$$

For compatibility, $q(x)$ needs to satisfy the condition

$$\int_\Omega q(x)dx = 0.$$

Define the spaces

$$L^2(\Omega) = \big\{ w : \int_\Omega |w(x)|^2 dx < \infty \big\},$$

$$H(\mathrm{div};\Omega) = \{v \in (L^2(\Omega))^3 : \nabla \cdot v \in L^2(\Omega)\},$$

$$W = \big\{ w \in L^2(\Omega) : \int_\Omega w(x)dx = 0 \big\},$$

$$V = \{v \in H(\mathrm{div};\Omega) : v \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

Then the mixed form of (A.1) for the pair $(u, p) \in V \times W$ is

(A.2a) $$(\nabla \cdot u, w) = (q, w), \qquad \forall w \in W,$$

(A.2b) $$(k^{-1}u, v) - (p, \nabla \cdot v) = 0, \qquad \forall v \in V,$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ or $(L^2(\Omega))^3$ inner product, as appropriate. This system has a unique solution [7].

For $0 < h < 1$, let $\mathcal{E}_h$ a partition of Ω into rectangular parallelepipeds. In $\mathcal{E}_h$ we need that adjacent elements completely share their common face. Then we introduce the Raviart-Thomas-Nedelec mixed space [21], [19] of lowest-order

$$V_h = \{v \in V : v|_E = (a_E^1 + a_E^2 x, a_E^3 + a_E^4 y, a_E^5 + a_E^6 z), a_E^i \in \Re, \forall E \in \mathcal{E}_h\},$$

$$W_h = \{w \in W : v|_E = b_E, b_E \in \Re, \forall E \in \mathcal{E}_h\}.$$

Then the mixed finite element solution of (A.2) is $(u_h, p_h) \in V_h \times W_h$ satisfying

(A.3a) $$(\nabla \cdot u_h, w) = (q, w), \qquad \forall w \in W_h,$$

(A.3b) $$(k^{-1}u_h, v) - (p_h, \nabla \cdot v) = 0, \qquad \forall v \in V_h.$$

Again, this system has a unique solution [7].

The linear system arising from (A.3) is a saddle point problem [7], which can be expensive to solve. One of useful numerical methods for solving this saddle point problem is the inexact Uzawa algorithm (see, e.g., [6], [16]). A more efficient approach was suggested by means of a nonmixed formulation. Namely, it has been shown that the mixed finite element method is equivalent to a modification of a nonconforming Galerkin method [2], [3], [9], [10]. The nonconforming method yields a symmetric and positive definite problem, which can be more easily solved.

References

1. S. Antoncev, *On the solvability of boundary value problems for degenerate two-phase porous flow equations*, in Russian, Dinamika Splošnoĭ Sredy Vyp. **10** (1972), 28–53.
2. T. Arbogast and Zhangxin Chen, *On the implementation of mixed methods as nonconforming methods for second-order elliptic problems*, Math. Comp. **64** (1995), 943–972.
3. D. N. Arnold and F. Brezzi, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, RAIRO Modél. Math. Anal. Numér. **19** (1985), 7–32.
4. J. Bear, *Dynamics of Fluids in Porous Media*, Dover, New York, 1972.
5. P. Binning, *Modeling unsaturated zone flow and contaminant transport in the air and water phases*, Ph. D. Thesis, Department of Civil Engineering and Operations Research, Princeton University, 1994.
6. J. Bramble, J. Pasciak, and A. Vassilev, *Analysis of the inexact Uzawa algorithm for saddle point problems*, ISC-94-09-MATH Technical Report, Texas A&M University.
7. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.
8. G. Chavent and J. Jaffré, *Mathematical Models and Finite Elements for Reservoir Simulation*, North-Holland, Amsterdam, 1978.
9. Zhangxin Chen, *Analysis of mixed methods using conforming and nonconforming finite element methods*, RAIRO Modél. Math. Anal. Numér. **27** (1993), 9–34.
10. Zhangxin Chen, *Equivalence between and multigrid algorithms for nonconforming and mixed methods for second order elliptic problems*, East-West J. Numer. Math. **4** (1996), to appear.
11. Zhangxin Chen, M. Espedal, and R. Ewing, *Continuous-time finite element analysis of multiphase flow in groundwater hydrology*, Applications of Mathematics (1995), to appear.
12. Zhangxin Chen and R. Ewing, *Fully-discrete finite element analysis of multiphase flow in groundwater hydrology*, ISC-95-04-MATH Technical Report, Texas A&M University.
13. Zhangxin Chen, R. Ewing, and M. Espedal, *Multiphase flow simulation with various boundary conditions*, Numerical Methods in Water Resources, Vol. 2, A. Peters, et als., eds., Kluwer Academic Publishers, Netherlands, 1994, pp. 925–932.
14. A. T. Corey, *Mathematics of Immiscible Fluids in Porous Media*, Water Resources Publication, Littleton, CO, 1986.
15. M. Delshad and G. A. Pope, *Comparison of the three-phase oil relative permeability models*, J. of Transport in Porous Media **4** (1989), 59–83.
16. H. Elman and G. Golub, *Inexact and preconditioned Uzawa algorithms for saddle point problems*, SIAM J. Numer. Anal. **31** (1994), 1645–1661.
17. R. Ewing, *Problems arising in the modeling of processes for hydrocarbon recovery*, The Mathematics of Reservoir Simulation, R. Ewing, Ed., SIAM, Philadelphia, 1983, pp. 3–34.
18. M. Leverett and W. Lewis, *Steady flow of gas-oil-water mixtures through unconsolidated sands*, Trans SPE of AIME **142** (1941), 107–116.

19. J. Nedelec, *Mixed finite elements in* $\Re^3$, Numer. Math. **35** (1980), 315–341.
20. D. W. Peaceman, *Fundamentals of Numerical Reservoir Simulation*, Elsevier, New York, 1977.
21. P.A. Raviart and J.M. Thomas, *A mixed finite element method for second order elliptic problems*, Lecture Notes in Math. **606**, Springer, Berlin, 1977, pp. 292–315.
22. H. L. Stone, *Estimation of three-phase relative permeability and residual oil data*, J. Can. Pet. Tech. **12** (1973), 53–67.

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