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Domain Splitting Algorithm for Mixed Finite Element Approximations to Parabolic Problems

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Abstract — In this paper we fromulate and study a domain decomposition algorithm for solving mixed finite element approximations to parabolic initial-boundary value problems. In contrast to the usual overlapping domain decomposition method this technique leads to noniterative algorithms, i.e. the subdomain problems are solved independently and the solution in the whole domain is obtained from the local solutions by restriction and simple averaging. The algorithm exploits the fact that the time discretization leads to an elliptic problem with a large positive coefficient in front of the zero order term. The solutions of such problems exhibit a boundary layer with thickness proportional to the square root of the time discretization parameter. Thus, any error in the boundary conditions will decay exponentially and a reasonable overlap will produce a sufficiently accurate method. We prove that the proposed algorithm is stable in L^2-norm and has the same accuracy as the implicit method.

Keywords. parabolic equations, domain decomposition, error estimate, stability, mixed finite elements, convergence.

1. INTRODUCTION

The main idea of the Schwarz alternating algorithm for solving partial differential equations is to divide the domain into a number of overlapping subdomains and solve a similar problem on each subdomain (alternatively or in parallel) with boundary information about the solution from the neighboring subdomains. This general idea has been applied to various elliptic problems (for a recent review on Schwarz methods we refer to Dryja and Widlund [11]). Since time discretization of parabolic problems leads to certain elliptic problems on the consecutive time level, domain decomposition algorithms can be applied to this class of problems as well. However, discretizations of parabolic equations have one very important feature: a large parameter in front of the zero order term. Exploring this property of the discretizations of time dependent problems, Kuznetsov proposed and studied in [14], [15], [16] a special class of domain decomposition algorithms (see also Blum, Lisky, and Rannacher [2], Rannacher and Zhou [17], Cai [5]).

We demonstrate the main idea of this method on the following problem. Let Ω denote a spatial domain in R^2 or R^3. Assume Ω has a piecewise uniformly smooth Lipschitz boundary, $\partial\Omega$. Assume that F, g, f, p^0, a, and b are smooth real-valued functions on the domain of definition, with a being a uniformly positive definite matrix and b

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nonnegative. For some $T > 0$, the function $p(x,t)$ satisfies

$$\begin{cases} p_t - \nabla \cdot (a\nabla p) + bp = F & \text{in } \Omega \times (0,T], \\ p = g & \text{on } \partial\Omega \times (0,T], \\ p(x,0) = p^0(x), & \text{in } \Omega. \end{cases} \tag{1.1}$$

We discretize the time variable by introducing $t_m = m\Delta t$ and write the implicit Euler approximation of (1.1):

$$\mathcal{L}p^m \equiv \frac{1}{\Delta t}p^m - \nabla \cdot a\nabla p^m + b^m p^m = \frac{1}{\Delta t}p^{m-1} + F^m, \quad \text{in } \Omega \quad p^m = g^m, \quad \text{on } \partial\Omega. \tag{1.2}$$

Here p^m denotes the approximation to $p(x,t_m)$ and $m = 1,2,\cdots$. Therefore, to find the solution at time level m we have to solve an elliptic problem with a large parameter in front of the zero order term. This in turn can be done by applying finite elements, finite differences, or finite volumes (see, e.g. [4], [8], [21]). In order to explain the idea of the domain-splitting algorithm, for a moment we shall assume that we can solve the elliptic problem (1.2) exactly. We also consider the problem

$$\mathcal{L}\bar{p}^m = \frac{1}{\Delta t}p^{m-1} + F^m, \quad \text{in } \Omega, \quad \bar{p}_m = \bar{g}^m, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\bar{g}^m$ is a perturbation of the boundary data g^m. Then the difference $e^m = p^m - \bar{p}^m$ will be a solution to the homogeneous equation with nonhomogeneous boundary data:

$$\mathcal{L}e^m = 0, \quad \text{in } \Omega, \quad e^m = g^m - \bar{g}^m \equiv \delta^m, \quad \text{on } \partial\Omega. \tag{1.4}$$

From the theory of elliptic problems with a large parameter $(\Delta t)^{-1}$ in front of the zero order term it is well known that the solution will exhibit a boundary layer of thickness proportional to $\sqrt{\Delta t}$. Namely, near the boundary the solution has the following behaviour $e^m(x) = O(\delta^m \exp(-\rho(x)/\sqrt{\Delta t}))$, where $\rho(x) = dist(x,\partial\Omega)$. Similarly, if Neumann boundary condition $\partial p^m/\partial n = \delta^m$ is prescribed on $\partial\Omega$, then the solution will have the following behavior near the boundary: $e^m(x) = O(\delta^m\sqrt{\Delta t}\exp(-\rho(x)/\sqrt{\Delta t}))$. For small Δt the solution will be almost zero outside the boundary layer. Model computations performed on PLTMG [1] that demonstrate the behavior of $e^m(x)$ in the case of Ω being a unit square for $h = 0.03$, $\Delta t = 0.005$, $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$, $b = 0$, and $\delta^m = -1$ are shown in Figure 1 and Figure 2 for Dirichlet and Neumann boundary conditions, respectively. These computations indicate that Neumann boundary conditions should lead to more accurate results since the error compared with the error of the Dirichlet problem is smaller by a factor of $\sqrt{\Delta t}$. Nevertheless, in both cases the decay of the boundary data has the same exponential rate.

These observations were the basis of the domain splitting algorithm, proposed and studied by Kuznetsov [14] (see also Blum, Lisky and Rannacher [2], Cai [5], Rannacher and Zhou [17]). This algorithm can be explained in the following simple manner. The domain Ω is covered by a fixed number of overlapping subdomains Ω_i^d with overlap of two adjacent subdomains no less than $2d$. Then the domain splitting algorithm reads as follow:

1. Initialize p^0; for $m = 1, ...$ perform the following two steps:

2. For each subdomain Ω_i^d, solve approximately the boundary value problem:

$$\begin{cases} \mathcal{L}p_i^m \equiv \frac{1}{\Delta t}p_i^m - \nabla \cdot a\nabla p_i^m + b^m p_i^m = \frac{1}{\Delta t}p^{m-1} + F^m, & x \in \Omega_i^d \\ lp_i^m = lp_i^{m-1} \quad \text{on } \partial\Omega_i^d. \end{cases} \tag{1.5}$$

where l is a linear operator associated with some boundary conditions. For example, l being the identity operator will correspond to Dirichlet boundary conditions, which will produce the method of Kuznetsov [14], and Blum, Lisky and Rannacher [2].

3. From the patch-wise solutions p_i^m, $i = 1, \cdots, N$ recover the solution p^m in Ω by restricting p_i^m to Ω_i and performing some averaging along the common boundary between two adjacent subregions.

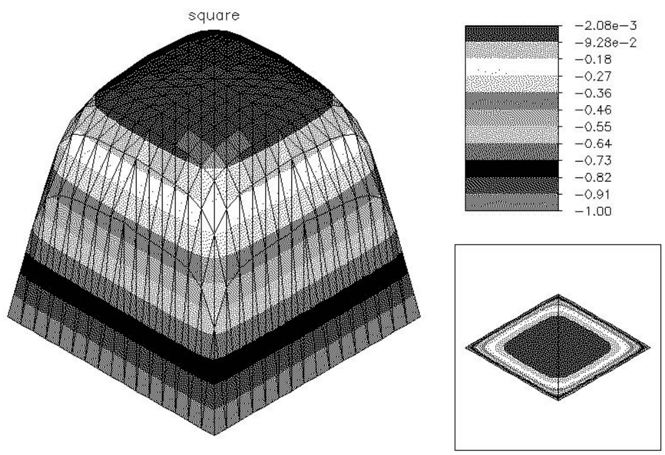

Figure 1. Unit square; Dirichlet boundary condition

This algorithm has one remarkable feature. The subdomain problems for level m are completely independent and can be solved in parallel on a distributed computer architecture. Synchronization is needed only once when computing the values of p^m along the interior boundaries. Furthermore, when advancing from t_{m-1} to t_m, one can use different time step-sizes in each subdomain Ω_i^d.

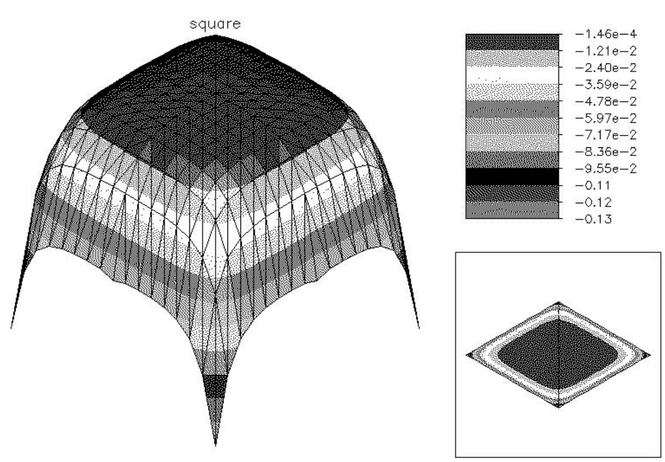

Figure 2. Unit square; Neumann boundary condition

Obviously, the choice $d = 0$ will lead to a meaningless algorithm. However, following Dawson, Du, and Dupont [9] (see, also, [10]) one can modify the boundary conditions in (1.5) by explicit approximation along the boundary $\partial\Omega_i^d$, using a much coarser mesh at level $m-1$ with step size $H >> h$. Dawson, Du, and Dupont [9] have shown that such a method is stable if $\Delta t = O(H^2)$ and has $O(h^2 + H^3 + \Delta t)$ asymptotic rate of convergence in the maximum norm. The choice $H = O(h^{2/3})$ will balance the terms in the error estimate and the scheme will be stable for $\Delta t = O(h^{4/3})$. This stability condition is much less restrictive than the Courant condition for a fully explicit scheme making the algorithm a good candidate for parallel realization.

If $d >> \sqrt{\Delta t}$ then the pollution due to an inaccurate boundary condition in (1.5) will be very small and the domain decomposition method should work quite well. Theoretically, this can be proved by using the general theory of local error estimates for singularly perturbed reaction-diffusion problems (see, e.g. Schatz and Wahlbin [19], lemma 2.2 and lemma 5.1). However, a larger overlap means more arithmetic work and more information to be exchanged between the adjacent subregions. This in turn relates to more intensive communications in the computer realization. The obvious question that arises is what is the minimal overlap that guarantees stability of the method and provides the same asymptotic order of accuracy as the global method (1.1) ?

The computational experiments performed on model problems by Blum, Lisky, and Rannacher in [2] show that this algorithm is stable and has the same accuracy as the global method (1.2) for an overlap that is proportional to the space-grid size. In these experiments d is typically $2h - 4h$. Unfortunately, the existing theory puts much more severe restriction on the overlap parameter d, than those observed in the computational experiments. Kuznetsov (e.g., [14],[16]) has shown that the domain decomposition scheme has the same accuracy as the standard implicit Galerkin scheme for $d \approx \sqrt{\Delta t}|\log \Delta t|$. This analysis is purely algebraic and is done for discretizations of convection-diffusion problems. Using analytic technique Blum, Lisky, and Rannacher [2] have proved that there is a constant γ such that $d \geq \gamma\sqrt{\Delta t}|\log \frac{d}{h}|$ is sufficient for the stability of the method (1.3). Also, the method will have the same asymptotic convergence rate as the global method (1.2). Actually, Blum, Lisky, and Rannacher [2] have considered more general domain splitting schemes, namely, Crank-Nicolson approximations in time, linear and quadratic extrapolation of the Dirichlet boundary data from levels $m-3$, $m-2$, and $m-1$, and general convex averaging along the subdomain interfaces.

Another interesting study of this algorithm applied to streamline diffusion finite element approximations to convection-diffusion problems has been done by Rannacher and Zhou in [17]. Again, the keys in their analysis are the decay property of the solution to a homogeneous equation with nonhomogeneous boundary data, strong L^2-stability, and an a priori estimate for the differences $p^m - p^{m-1}$.

The goal of this paper is to extend the formulation of the domain splitting algorithm to the mixed finite element approximations to parabolic problems and prove its stability and convergence. In section 2 we introduce the finite element spaces that are being used for second-order elliptic equations and formulate a backward in time Euler mixed finite element approximation to problem (1.1). Next, we introduce the domain splitting algorithm for this discretization. This algorithm is based on setting Neumann boundary conditions on the artificial internal boundaries of the subdomains, which in our opinion are more natural for the mixed finite element formulation. The main result of this paper is proved in Theorem 1.: under the condition $d \geq c_0\sqrt{\Delta t}|\log h\Delta t|$ the domain splitting algorithm is stable and is $O(\Delta t)$-close to the solution of the standard implicit mixed finite element method. The proof of Theorem 1. is the main result in this paper. Lemma 2. plays a key role in the proof, whereby a decay property of the the mixed finite element solution to a homogeneous reaction-diffusion equation with a small parameter in front of the second-order term and nonhomogeneous Neumann boundary conditions is established. Since the mixed finite elements approximate the pressure p and the velocity $\boldsymbol{u} = a\nabla p$ as independent variables, we had to first find the correct formulation of the decay property for both p and $\boldsymbol{u}$ and prove it for their mixed finite element approximations P and $\boldsymbol{U}$. This was the major theoretical difficulty we had to overcome. We note that the proof uses the inverse property of the finite element spaces and the weighted super-approximation property (3.1). Condition (2.14) on the overlapping parameter d plays an essential role in the proof. Also, in the inductive part of the proof we use the estimate (3.18) for the expression $\| \boldsymbol{U}_*^m - \boldsymbol{U}_*^{m-1} \|$. This estimate, formulated in Lemma 1. is derived in [7].

2. DOMAIN SPLITTING ALGORITHM

For the purpose of simplicity we shall consider the problem (1.1) with homogeneous Dirichlet boundary conditions, i.e. $g \equiv 0$. To describe the mixed variational form for (1.1), as usual, we introduce two Hilbert spaces. Let

$$W = L^2(\Omega), \qquad \boldsymbol{V} = \left\{ \boldsymbol{\varphi} \in L^2(\Omega)^2,\ \nabla \cdot \boldsymbol{\varphi} \in L^2(\Omega) \right\},$$

and the space $\boldsymbol{V}$ be equipped with norm $\| \boldsymbol{\varphi} \|_{\boldsymbol{V}} = (\| \boldsymbol{\varphi} \|^2 + \| \nabla \cdot \boldsymbol{\varphi} \|^2)^{1/2}$. The inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. And for the sake of simplicity, $(\cdot, \cdot)$ and $\| \cdot \|$ are also respectively be used as the inner product and norm in the product spaces $L^2(\Omega)^2$ or $L^2(\Omega)^3$.

Throughout this paper $H^i(\Omega)$ denotes the standard Sobolev spaces $W^{i,2}(\Omega)$, with $H^0(\Omega) = L^2(\Omega)$ and

$$H_0^1(\Omega) = \{ v \in H^1(\Omega),\ v = 0 \text{ in } \partial\Omega \}.$$

We also denote by $\boldsymbol{H}^i(\Omega)$ the vector analog of $H^i(\Omega)$.

Let $\boldsymbol{u} = a\nabla p$; then the pair $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ satisfies the following mixed variational equation:

$$\begin{cases} (p_t, \psi) - (\nabla \cdot \boldsymbol{u}, \psi) + (bp, \psi) = (F, \psi), & \forall\ \psi \in W, t \in (0, T], \\ (\alpha \boldsymbol{u}, \boldsymbol{\varphi}) + (\nabla \cdot \boldsymbol{\varphi}, p) = 0, & \forall\ \boldsymbol{\varphi} \in \boldsymbol{V}, t \in (0, T], \\ p(0) = p^0, \end{cases} \tag{2.1}$$

where $p_t = \partial p / \partial t$ and $\alpha = a^{-1}$. We note that the boundary condition $p = 0$ on $\partial\Omega$ is implicitly contained in (2.1).

For the spatial discretization of problem (2.1), let $\mathcal{T}_h = \{T\}$ be a quasi-uniform family of partitions of the domain Ω into closed triangles or rectangles with diameter $h > 0$, i.e. $\bar{\Omega} = \cup \{ T \in \mathcal{T}_h \}$. The mixed finite element spaces $W_h \subset W$ and $\boldsymbol{V}_h \subset \boldsymbol{V}$, are defined in a standard way. Examples of various particular spaces can be found in the monograph of Brezzi and Fortin [4], p. 125–128. We assume that

$$\nabla \cdot \boldsymbol{V}_h \subset W_h \tag{2.2}$$

and there exists a linear operator $\boldsymbol{\Pi}_h :\ \boldsymbol{V} \to \boldsymbol{V}_h$ such that

$$\nabla \cdot \boldsymbol{\Pi}_h = Q_h \nabla \cdot .$$

Here, the operator $Q_h: W \to W_h$ is the L^2 projection on W_h, i.e.,

$$(\psi - Q_h\psi, \psi_h) = 0, \quad \forall \psi \in W, \ \psi_h \in W_h.$$

Below, and throughout the paper, the letter C is used as a generic constant, which is independent of h, p, $\boldsymbol{u}$, etc.

We assume that the partition $\mathcal{T}_h$ and the finite element spaces W_h and $\boldsymbol{V}_h$ satisfy the inverse inequality

$$\|\boldsymbol{\varphi}_h\|_{\boldsymbol{H}^1(\Omega)} \leq Ch^{-1}\|\boldsymbol{\varphi}_h\|, \quad \forall\, \boldsymbol{\varphi}_h \in \boldsymbol{V}_h, \tag{2.3}$$

$$\|\psi_h\|_{H^1(\Omega)} \leq Ch^{-1}\|\psi_h\|, \quad \forall\, \psi_h \in W_h, \tag{2.4}$$

and the following approximation properties: there exists an integer $r \geq 0$ such that

$$\|\boldsymbol{\varphi} - \boldsymbol{\Pi}_h\boldsymbol{\varphi}\| \leq Ch^i\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^i(\Omega)}, \quad \forall\, \boldsymbol{\varphi} \in \boldsymbol{H}^i(\Omega),\ 1 \leq i \leq r+1, \tag{2.5}$$

$$\|\psi - Q_h\psi\| \leq Ch^i\|\psi\|_{H^i(\Omega)}, \quad \forall\, \psi \in H^i(\Omega),\ 0 \leq i \leq r+1. \tag{2.6}$$

After these preliminaries we state the backward in time Euler mixed finite element approximation in space of the problem (1.1). The global mixed finite element approximation $(P_*^m, \boldsymbol{U}_*^m) \in W_h \times \boldsymbol{V}_h$ to the exact solution $(p(\cdot,t_m), \boldsymbol{u}(\cdot,t_m)) \in W \times \boldsymbol{V}$ of (2.1) is the solution of the problem,

$$\begin{cases} \left(\dfrac{P_*^m - P_*^{m-1}}{\Delta t}, \Psi\right) - (\nabla\cdot\boldsymbol{U}_*^m - b^m P_*^m, \Psi) = (F^m, \Psi), & \forall\, \Psi \in W_h, \\ (\alpha^m \boldsymbol{U}_*^m, \boldsymbol{\Phi}) + (\nabla\cdot\boldsymbol{\Phi}, P_*^m) = 0, & \forall\, \boldsymbol{\Phi} \in \boldsymbol{V}_h, \\ P_*^0 = Q_h p^0 \in W_h. \end{cases} \tag{2.7}$$

Here, $m > 0$, $\Delta t > 0$ is the time-step size and $t_m = m\Delta t$. The initial approximation P_*^0 is taken as the L^2 projection in W_h of p^0 only for the sake of simplicity; any other choice that guarantees approximation can be used. For a function $\beta(t)$, we denote its value at $t = t_m$ by β^m.

Note that $\boldsymbol{U}_*^0$ is determined by P_*^0 through the relation

$$(\alpha^0\boldsymbol{U}_*^0, \boldsymbol{\Phi}) + (\nabla\cdot\boldsymbol{\Phi}, P_*^0) = 0, \quad \forall\, \boldsymbol{\Phi} \in \boldsymbol{V}_h. \tag{2.8}$$

To solve problem (2.7), we apply an overlapping domain decomposition method. To this end, let the domain Ω be divided into a finite number of polygonal subdomains Ω_i, $i = 1, \cdots, N$, each of which is a union of elements of T_h and has diameter $O(1)$. To each Ω_i, we associate a convex extension

$$\Omega_i^d = \cup\left\{T \in \mathcal{T}_h : \ \mathrm{dist}(T, \Omega_i) \leq d\right\},$$

where $d \geq h$ is the overlap width. Further, let

$$\Gamma_i = \partial\Omega_i^d \backslash \partial\Omega, \quad \Gamma_{ij} = \Gamma_{ji} = \bar{\Omega}_i \cap \bar{\Omega}_j,$$

denote the interior boundary of subdomain Ω_i^d and the common interior boundary of subdomains Ω_i and Ω_j, respectively.

For each subdomain Ω_i^d, we denote the restrictions of W_h and $\boldsymbol{V}_h$ to Ω_i^d by $W_h(\Omega_i^d)$ and $\boldsymbol{V}_h(\Omega_i^d)$, respectively. And let

$$\boldsymbol{V}_h^0(\Omega_i^d) = \left\{\boldsymbol{v} \in \boldsymbol{V}_h(\Omega_i^d) : \ \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma_i} = 0\right\},$$

where $\boldsymbol{n}$ denotes the exterior unit normal direction.

Now, we describe the overlapping domain splitting algorithm for problem (2.7).

Step 0. We choose some approximation $P^0 \in W_h$ to the initial value $p(0)$, e.g., its L^2-projection in W_h. The initial approximation $\boldsymbol{U}^0$ for $\boldsymbol{u}(0)$ is determined by (2.8).

If approximations $(P^i, \boldsymbol{U}^i) \in W_h \times \boldsymbol{V}_h$ to $(p^i, \boldsymbol{u}^i) \equiv (p(\cdot, t_i), \boldsymbol{u}(\cdot, t_i)) \in W \times \boldsymbol{V}$, have been calculated for all $0 \le i < m$, then $(P^m, \boldsymbol{U}^m) \in W_h \times \boldsymbol{V}_h$ is determined through the following steps:

Step 1. On each artificial interior boundary $\Gamma_i \neq \emptyset$, we calculate the normal component $\bar{\boldsymbol{U}} \cdot \boldsymbol{n}$ by constant extrapolation in time from the previous time level:

$$\bar{\boldsymbol{U}}^m \cdot \boldsymbol{n} = \boldsymbol{U}^{m-1} \cdot \boldsymbol{n}. \tag{2.9}$$

Step 2. In each subdomain Ω_i^d, $i = 1, ..., N$, the local approximations $(P_i^m, \boldsymbol{U}_i^m) \in W_h(\Omega_i^d) \times \boldsymbol{V}_h(\Omega_i^d)$ are computed independently by solving the following local mixed finite element problems:

$$\begin{cases} \left(\dfrac{P_i^m - P^{m-1}}{\Delta t}, \Psi\right)_{\Omega_i^d} - (\nabla \cdot \boldsymbol{U}_i^m - b^m P_i^m, \Psi)_{\Omega_i^d} = (f^m, \Psi)_{\Omega_i^d}, & \forall\ \Psi \in W_h(\Omega_i^d), \\ (\alpha^m \boldsymbol{U}_i^m, \Phi)_{\Omega_i^d} + (\nabla \cdot \Phi, P_i^m)_{\Omega_i^d} = 0, & \forall\ \Phi \in \boldsymbol{V}_h^0(\Omega_i^d), \\ \boldsymbol{U}_i^m \cdot \boldsymbol{n}|_{\Gamma_i} = \bar{\boldsymbol{U}}^m \cdot \boldsymbol{n}\Big|_{\Gamma_i}. \end{cases} \tag{2.10}$$

Step 3. The global approximation $(P^m, \boldsymbol{U}^m) \in W_h \times \boldsymbol{V}_h$ is formed from the restrictions of the patch-wise solutions $(P_i^m, \boldsymbol{U}_i^m) \in W_h(\Omega_i^d) \times \boldsymbol{V}_h(\Omega_i^d)$ on Ω_i:

$$P^m|_{\Omega_i} = P_i^m|_{\Omega_i}, \tag{2.11}$$

$$\boldsymbol{U}^m|_{\Omega_i \setminus \partial\Omega_i} = \boldsymbol{U}_i^m|_{\Omega_i \setminus \partial\Omega_i}, \tag{2.12}$$

and by averaging of the normal components of the velocity $\boldsymbol{U}_i^m$ along the artificial interior boundaries:

$$\boldsymbol{U}^m \cdot \boldsymbol{n} = \frac{1}{2}\left(\boldsymbol{U}_i^m \cdot \boldsymbol{n} + \boldsymbol{U}_j^m \cdot \boldsymbol{n}\right), \qquad \text{on } \Gamma_{ij}. \tag{2.13}$$

The main result in this paper is a proof of the stability of the domain decomposition method. This result is established in the following theorem:

THEOREM 1. *Let the solution of problem (1.1) be sufficiently smooth and let the overlapping parameter d satisfy the condition*

$$d \ge c_0 \sqrt{\Delta t} |\log(h\Delta t)| \quad \textit{for some constant} \quad c_0 > 0. \tag{2.14}$$

Then the domain decomposition algorithm (2.9) – (2.13) is stable in the L^2-norm and the following estimate holds true:

$$\| P_*^m - P^m \|^2 + \Delta t \, \| \boldsymbol{U}_*^m - \boldsymbol{U}^m \|^2 \le C(\Delta t)^3. \tag{2.15}$$

This theorem says that the domain splitting algorithm is $O(\Delta t)$-close in L^2-norm to the solution of the global method (2.7). In order to estimate the error of the algorithm we can use the existing estimates for $\| p(t_m) - P^m \|$ and $\| u(t_m) - \boldsymbol{U}^m \|$ Chen and Lazarov [7]. Namely, the following estimate has (see, e.g. [7]):

LEMMA 1. *For $m \geq 1$ let $(p(t_m), \boldsymbol{u}(t_m)) \equiv (p^m, \boldsymbol{u}^m) \in W_h \times \boldsymbol{V}_h$ and $(P_*^m, \boldsymbol{U}_*^m) \in W_h \times \boldsymbol{V}_h$ be the solutions of problems (1.1) and (2.7), respectively. If the solution to problem (1.1) is sufficiently smooth, then*

$$\| p^m - P_*^m \| + \| \boldsymbol{u}^m - \boldsymbol{U}_*^m \| \leq C(h^{r+1} + \Delta t). \tag{2.16}$$

Combining the results of Theorem 1. and Lemma 1. we conclude that the domain splitting algorithm will produce an approximate solution that is as accurate as the solution of the global method (2.7).

Theorem 1. will be the subject of our study in section 3.

3. STABILITY ESTIMATE

In this section, we give a proof of Theorem 1. from Section 2. The proof is based on the decay property of the mixed finite element solution of a singularly perturbed elliptic equation with nonhomogeneous Neumann boundary data.

For our analyisis $\boldsymbol{V}_h(\Omega_i^d)$ has to satisfy the so-called weighted super-approximation property. First, we define $\rho(x) = \mathrm{dist}(x, \Gamma_i)$ and $\phi(x) = e^{\gamma\rho(x)/\sqrt{\Delta t}}$, where $\gamma \in (0, 1]$ is a constant which will be determined later. We say that the finite element space $\boldsymbol{V}_h(\Omega_i^d)$ satisfies the weighted super-approximation property if there is a constant C independent of h and such that for any $\boldsymbol{W} \in \boldsymbol{V}_h(\Omega_i^d)$ the following inequality is valid:

$$\| \phi^{-1/2}(\phi \boldsymbol{W} - \tilde{\boldsymbol{W}}) \|_{L^2(\Omega_i^d)} \leq C\gamma \, \| \phi^{1/2} \boldsymbol{W} \|_{L^2(\Omega_i^d)}, \tag{3.1}$$

where $\tilde{\boldsymbol{W}}$ is the quasi-interpolant of $\phi \boldsymbol{W}$ in $\boldsymbol{V}_h(\Omega_i^d)$, i.e. $\tilde{\boldsymbol{W}} \equiv \boldsymbol{\Pi}_h(\phi \boldsymbol{W}) \in \boldsymbol{V}_h(\Omega_i^d)$.

The L^p-error analysis for $1 \leq p \leq \infty$ of conforming finite element approximations often uses this type of property (see, e.g. [12], [20]). We are not aware of a direct proof of this property based only on the definition of mixed finite element spaces $\boldsymbol{V}_h$ and W_h in section 1. One can verify this super-approximation property for each particular pair of spaces $\boldsymbol{V}_h$ and W_h provided that $\boldsymbol{\Pi}_h$ is defined element by element. To be specific we consider the Raviart-Thomas rectangular element of order zero [18]. In the same way one can treat the triangular Raviart-Thomas element of order zero. Let $K \in \mathcal{T}_h$ and l_i, $i = 1, 2, 3, 4$ be the edges of K. For $\phi \boldsymbol{W}$ we define $\boldsymbol{\Pi}_K(\phi \boldsymbol{W}) \in \boldsymbol{V}_h(K)$ by

$$\int_{l_i} (\phi \boldsymbol{W} - \boldsymbol{\Pi}_K(\phi \boldsymbol{W})) \cdot \boldsymbol{n} ds = 0, \quad i = 1, 2, 3, 4. \tag{3.2}$$

Then $\boldsymbol{\Pi}_h(\phi \boldsymbol{W}) \in \boldsymbol{V}_h(\Omega_i^d)$ is given by

$$\boldsymbol{\Pi}_h(\phi \boldsymbol{W})|_K = \boldsymbol{\Pi}_K(\phi \boldsymbol{W}), \quad \forall K \in \mathcal{T}_h.$$

Obviously, $\phi \boldsymbol{W} = \boldsymbol{W}$ on the boundary $\partial\Omega_i^d$. This implies that $\boldsymbol{\Pi}_K(\phi \boldsymbol{W}) \cdot \boldsymbol{n} = \boldsymbol{W} \cdot \boldsymbol{n}$ on $\partial\Omega_i^d$. Due to (3.2) $\tilde{\boldsymbol{W}}$ satisfies the following identity:

$$(\nabla \cdot (\phi \boldsymbol{W} - \tilde{\boldsymbol{W}}), \Psi)_{\Omega_i^d} = 0, \quad \forall \, \Psi \in W_h(\Omega_i^d), \quad \tilde{\boldsymbol{W}} - \boldsymbol{W} \in \boldsymbol{V}_h^0(\Omega_i^d).$$

Once $\tilde{\boldsymbol{W}}$ is constructed locally, then the super-approximation property (3.1) is derived by using the locality of the operator $\boldsymbol{\Pi}_h$ and the following inequality

$$\sup_{x \in K} \phi(x) / \inf_{x \in K} \phi(x) \leq C_0, \quad \forall K \in \mathcal{T}_h, \tag{3.3}$$

with a constant C_0 independent of h. Indeed, let $\xi \in K$, where $K \in \mathcal{T}_h$ and denote by $\phi_0 = \phi(\xi)$. Then

$$\begin{aligned} ||\phi \boldsymbol{W} - \boldsymbol{\Pi}_K(\phi \boldsymbol{W})||_{L^2(K)} &\leq ||(\phi - \phi_0)\boldsymbol{W} - \boldsymbol{\Pi}_K((\phi - \phi_0)\boldsymbol{W})||_{L^2(K)} \\ &\leq Ch \,\|\, (\phi - \phi_0)\boldsymbol{W} \,\|_{H^1(K)} \\ &\leq Ch(\| \nabla \phi \boldsymbol{W} \|_{L^2(K)} + \max|(\phi - \phi_0)| \, \| \boldsymbol{W} \|_{H^1(K)}) \\ &\leq C\frac{\gamma h}{\sqrt{\Delta t}} \sup_{x \in K} \phi(x) \, \| \boldsymbol{W} \|_{L^2(K)} \, . \end{aligned}$$

Here we have used the fact that $\nabla \phi$ is a piece-wise smooth vector function and there is a positive constant C, independent of h, such that $|\nabla \phi| \leq \dfrac{C\gamma}{\sqrt{\Delta t}}\phi$ with $|.|$ denoting the Euclidean norm. Then using (3.3) and assuming that $h^2 \leq C\Delta t$ we get

$$\| \phi^{-1/2}(\phi \boldsymbol{W} - \tilde{\boldsymbol{W}}) \|^2_{L^2(K)} \leq C\gamma^2 \, \| \phi^{1/2} \boldsymbol{W} \|^2_{L^2(K)} \, . \tag{3.4}$$

The required super-approximation property (3.1) follows by summing (3.4) over all finite elements $K \in \mathcal{T}_h$. The analysis of all other mixed finite elements which contain the lowest order Raviart-Thomas rectangles or triangles applies the same argument and the technique developed by Dupont and Scott [12]. A super-approximation property of this type was used by Wang [22] in the convergence and superconvergence analysis of mixed finite element approximations to second order elliptic problems.

The decay property is discussed and established in the following lemma:

LEMMA 2. **(decay property lemma)** *Let* $(Q, \boldsymbol{W}) \in W_h(\Omega_i^d) \times \boldsymbol{V}_h(\Omega_i^d)$ *satisfy*

$$\begin{cases} (Q, \Psi)_{\Omega_i^d} - \Delta t \, (\nabla \cdot \boldsymbol{W} - bQ, \Psi)_{\Omega_i^d} = 0, & \forall \ \Psi \in W_h(\Omega_i^d), \\ (\alpha \boldsymbol{W}, \boldsymbol{\Phi})_{\Omega_i^d} + (\nabla \cdot \boldsymbol{\Phi}, Q)_{\Omega_i^d} = 0, & \forall \ \boldsymbol{\Phi} \in \boldsymbol{V}_h^0(\Omega_i^d), \\ \boldsymbol{W} \cdot \boldsymbol{n}|_{\Gamma_i} = \boldsymbol{G} \cdot \boldsymbol{n}|_{\Gamma_i} \, , \end{cases} \tag{3.5}$$

with $\boldsymbol{G} \in \boldsymbol{V}_h$. *Then, there exist positive constants* C *and* γ, *such that*

$$\frac{1}{\Delta t} \, \| Q \|^2_{L^2(\Omega_i)} + \| \boldsymbol{W} \|^2_{L^2(\Omega_i)} \leq Ce^{-\gamma d/\sqrt{\Delta t}}(\| \boldsymbol{G} \|^2_{L^2(\Omega_i^d)} + \Delta t \, \| \nabla \cdot \boldsymbol{G} \|^2_{L^2(\Omega_i^d)}), \tag{3.6}$$

where $d = \operatorname{dist}(\Omega_i, \Gamma_i)$.

Proof : Since $\rho \geq d$ on Ω_i, we have

$$\frac{1}{\Delta t} \, \| Q \|^2_{L^2(\Omega_i)} + \| \boldsymbol{W} \|^2_{L^2(\Omega_i)} \leq e^{-\gamma d/\sqrt{\Delta t}} \left(\frac{1}{\Delta t}(\phi Q, Q)_{\Omega_i^d} + (\phi \boldsymbol{W}, \boldsymbol{W})_{\Omega_i^d} \right) . \tag{3.7}$$

We shall bound the right-hand side of (3.7). Let $\tilde{Q}$ denote the L^2 projection of ϕQ in $W_h(\Omega_i^d)$. From the first equation of (3.5), it follows that

$$\begin{aligned} \frac{1}{\Delta t}(\phi Q, Q)_{\Omega_i^d} &= \frac{1}{\Delta t}(\tilde{Q}, Q)_{\Omega_i^d} = (\nabla \cdot \boldsymbol{W} - bQ, \tilde{Q})_{\Omega_i^d} = (\nabla \cdot \boldsymbol{W}, \phi Q)_{\Omega_i^d} - (bQ, \tilde{Q})_{\Omega_i^d} \\ &= (\nabla \cdot (\phi \boldsymbol{W}), Q)_{\Omega_i^d} - (\nabla \phi \cdot \boldsymbol{W}, Q)_{\Omega_i^d} - (bQ, \tilde{Q})_{\Omega_i^d}. \end{aligned} \tag{3.8}$$

From (3.8) and the second equation of (3.5), and taking into account that $\tilde{\boldsymbol{W}} - \boldsymbol{G} \in \boldsymbol{V}_h^0(\Omega_i^d)$, we get

$$\begin{aligned} \frac{1}{\Delta t}(\phi Q, Q)_{\Omega_i^d} &= (\nabla \cdot (\tilde{\boldsymbol{W}} - \boldsymbol{G}), Q)_{\Omega_i^d} - (\nabla \phi \cdot \boldsymbol{W}, Q)_{\Omega_i^d} + (\nabla \cdot \boldsymbol{G}, Q)_{\Omega_i^d} - (bQ, \tilde{Q})_{\Omega_i^d} \\ &= -(\alpha \boldsymbol{W}, \tilde{\boldsymbol{W}} - \boldsymbol{G})_{\Omega_i^d} - (\nabla \phi \cdot \boldsymbol{W}, Q)_{\Omega_i^d} + (\nabla \cdot \boldsymbol{G}, Q)_{\Omega_i^d} - (bQ, \tilde{Q})_{\Omega_i^d}, \end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{\Delta t}(\phi Q, Q)_{\Omega_i^d} + \alpha_0(\phi \boldsymbol{W}, \boldsymbol{W})_{\Omega_i^d} \;\; &\le \;\; (\alpha \boldsymbol{W}, \phi \boldsymbol{W} - \tilde{\boldsymbol{W}})_{\Omega_i^d} - (\nabla\phi \cdot \boldsymbol{W}, Q)_{\Omega_i^d} \\ &\quad + (\nabla \cdot \boldsymbol{G}, Q)_{\Omega_i^d} + (\alpha \boldsymbol{W}, \boldsymbol{G})_{\Omega_i^d} - (bQ, \tilde{Q})_{\Omega_i^d}. \quad (3.9)\end{aligned}$$

Now we estimate the terms on the right hand side of (3.9). For the first term we use the weighted super-approximation property (3.1) to get:

$$\begin{aligned}(\alpha \boldsymbol{W}, \phi \boldsymbol{W} - \tilde{\boldsymbol{W}})_{\Omega_i^d} \;\; &\le \;\; C \,\|\phi^{1/2}\boldsymbol{W}\|_{L^2(\Omega_i^2)} \cdot \|\phi^{-1/2}(\phi \boldsymbol{W} - \tilde{\boldsymbol{W}})\|_{L^2(\Omega_i^d)} \\ &\le \;\; C\gamma(\phi \boldsymbol{W}, \boldsymbol{W})_{\Omega_i^d}. \quad (3.10)\end{aligned}$$

Using the estimate $|\nabla\phi| \le \dfrac{C\gamma}{\sqrt{\Delta t}}\phi$ we get the following bound for the second term in the right hand side of (3.9):

$$\Big|(\nabla\phi \cdot \boldsymbol{W}, Q)_{\Omega_i^d}\Big| \le \frac{C\gamma}{\Delta t}(\phi Q, Q)_{\Omega_i^d} + C\gamma(\phi \boldsymbol{W}, \boldsymbol{W})_{\Omega_i^d}. \quad (3.11)$$

The next two terms in (3.9) are estimated in the following manner:

$$(\nabla \cdot \boldsymbol{G}, Q)_{\Omega_i^d} \le \frac{\epsilon}{\Delta t}(\phi Q, Q)_{\Omega_i^d} + \frac{\Delta t}{4\epsilon}(\phi^{-1}\nabla \cdot \boldsymbol{G}, \nabla \cdot \boldsymbol{G})_{\Omega_i^d},$$

$$(\alpha \boldsymbol{W}, \boldsymbol{G})_{\Omega_i^d} \le \epsilon(\phi \boldsymbol{W}, \boldsymbol{W})_{\Omega_i^d} + \frac{C}{4\epsilon}(\phi^{-1}\boldsymbol{G}, \boldsymbol{G})_{\Omega_i^d},$$

where $\epsilon > 0$ is arbitrary.

Finally, for the last term in (3.9) we apply the local inverse inequality for $Q \in W_h(K)$ and $K \in \mathcal{T}_h$:

$$h||\nabla Q||_{L^2(K)} \le C||Q||_{L^2(K)}$$

and get

$$-(bQ, \tilde{Q})_{\Omega_i^d} \le -(bQ, \tilde{Q} - Q)_{\Omega_i^d} \le \frac{C}{\Delta t}(\Delta t + \gamma h^2)(\phi Q, Q)_{\Omega_i^d}.$$

We use these estimates for the terms in the right hand side of (3.9). Next, we fix γ sufficiently small, then choose $\Delta t \le \gamma$, and note that $\phi^{-1} \le 1$ in Ω_i^d. This completes the proof of the decay property lemma.

Proof of Theorem 1.: We compare $(P^m, \boldsymbol{U}^m)$ with the pair $(\tilde{P}^m, \tilde{\boldsymbol{U}}^m)$, which is the solution of the following auxiliary problem:

$$\begin{cases} \left(\dfrac{\tilde{P}^m - P^{m-1}}{\Delta t}, \Psi\right) - \left(\nabla \cdot \tilde{\boldsymbol{U}}^m - b^m \tilde{P}^m, \Psi\right) = (F^m, \Psi), & \forall\; \Psi \in W_h, \\ (\alpha \tilde{\boldsymbol{U}}^m, \boldsymbol{\Phi}) + (\nabla \cdot \boldsymbol{\Phi}, \tilde{P}^m) = 0, & \forall\; \boldsymbol{\Phi} \in \boldsymbol{V}_h. \end{cases} \quad (3.12)$$

First, for any $\varepsilon \in (0, 1]$, we write the inequalities

$$\begin{aligned}\|\boldsymbol{U}^m - \boldsymbol{U}_*^m\|^2 \;\; &\le \;\; \left(1 + \frac{1}{\varepsilon}\right)\|\boldsymbol{U}^m - \tilde{\boldsymbol{U}}^m\|^2 + (1+\varepsilon)\,\|\tilde{\boldsymbol{U}}^m - \boldsymbol{U}_*^m\|^2 \\ &\le \;\; \frac{C}{\varepsilon}\sum_{i=1}^{N}\|\boldsymbol{U}_i^m - \tilde{\boldsymbol{U}}^m\|^2_{L^2(\Omega_i)} + (1+\varepsilon)\,\|\tilde{\boldsymbol{U}}^m - \boldsymbol{U}_*^m\|^2,\end{aligned}$$

$$\|P^m - P_*^m\|^2 \le \frac{C}{\varepsilon}\sum_{i=1}^{N}\|P_i^m - \tilde{P}^m\|^2_{L^2(\Omega_i)} + (1+\varepsilon)\,\|\tilde{P}^m - P_*^m\|^2\,.$$

Combining these two estimates and applying Lemma 2. to $Q = P_i^m - \tilde{P}^m$ and $\boldsymbol{W} = \boldsymbol{U}_i^m - \tilde{\boldsymbol{U}}^m$ we get:

$$
\begin{aligned}
&\| P^m - P_*^m \|^2 + \alpha_0 \Delta t \, \| \boldsymbol{U}^m - \boldsymbol{U}_*^m \|^2 \\
&\quad \le (1+\varepsilon) \left(\| \tilde{P}^m - P_*^m \|^2 + \alpha_0 \Delta t \, \| \tilde{\boldsymbol{U}}^m - \boldsymbol{U}_*^m \|^2 \right) \\
&\qquad + \frac{C\Delta t}{\varepsilon} e^{-\gamma d/\sqrt{\Delta t}} \sum_{i=1}^{N} \left(\| \bar{\boldsymbol{U}}^{m-1} - \tilde{\boldsymbol{U}}^m \|_{L(\Omega_i^d)}^2 + \Delta t \, \| \nabla \cdot (\bar{\boldsymbol{U}}^{m-1} - \tilde{\boldsymbol{U}}^m \|_{L(\Omega_i^d)}^2 \right) \\
&\quad \le (1+\varepsilon) \left(\| \tilde{P}^m - P_*^m \|^2 + \alpha_0 \Delta t \, \| \tilde{\boldsymbol{U}}^m - \boldsymbol{U}_*^m \|^2 \right) \\
&\qquad + \frac{C\Delta t}{\varepsilon} e^{-\gamma d/\sqrt{\Delta t}} (1 + \frac{\Delta t}{h^2}) \left(\| \boldsymbol{U}^{m-1} - \boldsymbol{U}_*^{m-1} \|^2 + \| \boldsymbol{U}_*^{m-1} - \boldsymbol{U}_*^m \|^2 + \| \boldsymbol{U}_*^m - \tilde{\boldsymbol{U}}^m \|^2 \right).
\end{aligned} \tag{3.13}
$$

Next, we show that there exists a constant $\lambda > 0$ such that

$$
\| \tilde{P}^m - P_*^m \|^2 + \alpha_0 \Delta t \, \| \tilde{\boldsymbol{U}}^m - \boldsymbol{U}_*^m \|^2 \le \frac{1}{1+\lambda\Delta t} \, \| P_*^{m-1} - P^{m-1} \|^2 . \tag{3.14}
$$

Indeed, subtracting (2.7) from (3.12) and denoting $\boldsymbol{W}^m = \tilde{\boldsymbol{U}}^m - \boldsymbol{U}_*^m$ and $Q^m = \tilde{P}^m - P_*^m$ we get

$$
\left\{
\begin{array}{l}
(Q^m, \Psi) - \Delta t (\nabla \cdot \boldsymbol{W}^m - b^m Q^m, \Psi) = (P_*^{m-1} - P^{m-1}, \Psi), \quad \forall \ \Psi \in \boldsymbol{W}_h, \\
(\alpha \boldsymbol{W}^m, \boldsymbol{\Phi}) + (\nabla \cdot \boldsymbol{\Phi}, Q^m) = 0.
\end{array}
\right. \tag{3.15}
$$

Then, by taking $\Psi = Q^m$ and $\boldsymbol{\Phi} = \boldsymbol{W}^m$ in (3.15), using the fact that b is nonnegative, and $\alpha \ge \alpha_0$, it follows that

$$
\begin{aligned}
&\| Q^m \|^2 + \alpha_0 \Delta t \, \| \boldsymbol{W}^m \|^2 \\
&\quad \le (Q^m, Q^m) - \Delta t (\nabla \cdot \boldsymbol{W}^m - b^m Q^m, Q^m) = (P_*^{m-1} - P^{m-1}, Q^m) \\
&\quad \le \frac{1}{(1+\lambda\Delta t)^{1/2}} \, \| P_*^{m-1} - P^{m-1} \| \cdot (1+\lambda\Delta t)^{1/2} \, \| Q^m \| \\
&\quad \le \frac{1}{2(1+\lambda\Delta t)} \, \| P_*^{m-1} - P^{m-1} \|^2 + \frac{1}{2} \left(\| Q^m \|^2 + \lambda \Delta t \, \| Q^m \|^2 \right) \\
&\quad \le \frac{1}{2(1+\lambda\Delta t)} \, \| P_*^{m-1} - P^{m-1} \|^2 + \frac{1}{2} \left(\| Q^m \|^2 + C\lambda \Delta t \, \| \boldsymbol{W}^m \|^2 \right),
\end{aligned} \tag{3.16}
$$

where in the last step we have used the inequality $\| Q^m \| \le C \, \| \boldsymbol{W}^m \|$. Choosing λ sufficiently small, we obtain the desired estimate (3.14). Now, to complete the proof, we employ an induction argument. Since $\boldsymbol{U}^0 - \boldsymbol{U}_*^0 = 0$ and $P^0 - P_*^0 = 0$, we assume that

$$
\| P^{m-1} - P_*^{m-1} \|^2 + \alpha_0 \Delta t \, \| \boldsymbol{U}^{m-1} - \boldsymbol{U}_*^{m-1} \|^2 \le M (\Delta t)^3. \tag{3.17}
$$

Here, the constant M is assumed to be the same constant as in the estimate

$$
\| \boldsymbol{U}_*^m - \boldsymbol{U}_*^{m-1} \|^2 \le M (\Delta t)^2 \tag{3.18}
$$

established by Chen and Lazarov in [7].

Substituting (3.14) into (3.13), we obtain

$$
\begin{aligned}
&\| P^m - P_*^m \|^2 + \alpha_0 \Delta t \, \| \boldsymbol{U}^m - \boldsymbol{U}_*^m \|^2 \\
&\le \frac{1}{1+\lambda\Delta t}\left(1+\varepsilon+\frac{Ce^{-\gamma d/\sqrt{\Delta t}}}{\varepsilon\alpha_0}(1+\frac{\Delta t}{h^2})\right) \| P_*^{m-1} - P^{m-1} \|^2 \\
&\quad + \frac{C\Delta t}{\varepsilon} e^{-\gamma d/\sqrt{\Delta t}}(1+\frac{\Delta t}{h^2}) \left(\| \boldsymbol{U}^{m-1} - \boldsymbol{U}_*^{m-1} \|^2 + \| \boldsymbol{U}_*^{m-1} - \boldsymbol{U}_*^m \|^2 \right).
\end{aligned}
\tag{3.19}
$$

First we set

$$
\varepsilon = \frac{\lambda\Delta t}{2}, \qquad \varepsilon_1 = \frac{Ce^{-\gamma L}}{\varepsilon}(1+\frac{\Delta t}{h^2}), \quad L = \frac{d}{\sqrt{\Delta t}}
$$

and then, choose L sufficiently large so that

$$
\frac{1+\varepsilon+\dfrac{\varepsilon_1}{\alpha_0}}{1+\lambda\Delta t} < 1-\varepsilon_1, \quad \text{and} \quad L \ge C_0 \log\frac{1}{h\Delta t} \quad \text{for some } C_0.
$$

Now, from (3.19) and for sufficiently small ε_1, we get

$$
\begin{aligned}
&\| P^m - P_*^m \|^2 + \alpha_0 \Delta t \, \| \boldsymbol{U}^m - \boldsymbol{U}_*^m \|^2 \\
&\le \varepsilon_1 \Delta t \, \| \boldsymbol{U}_*^m - \boldsymbol{U}_*^{m-1} \|^2 \\
&\quad + (1-\varepsilon_1)\left(\| P^{m-1} - P_*^{m-1} \|^2 + \frac{\varepsilon_1 \Delta t}{1-\varepsilon_1} \| \boldsymbol{U}^{m-1} - \boldsymbol{U}_*^{m-1} \|^2 \right) \\
&\le \varepsilon_1 \Delta t \, \| \boldsymbol{U}_*^m - \boldsymbol{U}_*^{m-1} \|^2 \\
&\quad + (1-\varepsilon_1)\left(\| P^{m-1} - P_*^{m-1} \|^2 + \alpha_0 \Delta t \, \| \boldsymbol{U}^{m-1} - \boldsymbol{U}_*^{m-1} \|^2 \right).
\end{aligned}
$$

Now, using the induction assumption (3.17) and (3.18), we obtain

$$
\| P^m - P_*^m \|^2 + \alpha_0 \Delta t \, \| \boldsymbol{U}^m - \boldsymbol{U}_*^m \|^2 \le M(\Delta t)^3,
$$

which completes the proof of Theorem 1..

4. REMARKS AND CONCLUSIONS

The theory developed in this paper can be applied also to Crank-Nicolson mixed finite element approximations of parabolic problems. In this case the decay property of the mixed method will be the same. However, we need a theory for the error estimate of the global method. To our knowledge the theory of the mixed finite element methods has not covered this interesting case. Nevertheless, the necessary estimates should be derived using the technique developed by Thomee [21], Johnson and Thomee [13], and Chen and Lazarov [7].

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