A Wavelet–Galerkin Method for the Stokes Equations

Wolfgang Dahmen Angela Kunoth* Karsten Urban†

August 17, 1995 — Revised version

Abstract

The purpose of this paper is to investigate Galerkin schemes for the Stokes–equations based on a suitably adapted multiresolution analysis. In particular, it will be shown that techniques developed in connection with shift–invariant refinable spaces give rise to trial spaces of any desired degree of accuracy satisfying the Ladyšenskaja–Babuška–Brezzi condition for any spatial dimension. Moreover, in the time dependent case efficient preconditioners for the Schur complements of the discrete systems of equations can be based on corresponding stable multiscale decompositions. The results are illustrated by some concrete examples of adapted wavelets and corresponding numerical experiments.

In dieser Arbeit werden Galerkin-Verfahren für das Stokes-Problem untersucht, die auf speziell angepaßten Multiresolution-Ansätzen beruhen. Insbesondere wird gezeigt, daß gewisse Konstruktionsprinzipien für Wavelets auf gleichförmigen Gittern für jede Raumdimension und beliebige gewünschte Exaktheitsordnung auf Paare von Ansatzräumen führen, die die Ladyšenskaja–Babuška–Brezzi–Bedingung erfüllen. Darüber hinaus ergeben sich auch im instationären Fall aus den entsprechenden stabilen Multiskalenzerlegungen effiziente Vorkonditionierer für die Schurkomplemente entsprechenden Systemmatrizen. Die Ergebnisse werden anhand einiger konkreter Realisierungen und numerischer Tests illustriert.

Key words: Saddle point problems, LBB–condition, multiresolution analysis, wavelets, time dependent problems, Schur complements, preconditioning.
AMS subject classification: 15A12, 35Q30, 65F35, 65N30, 41A17, 41A63.

1 Introduction

During the past few years very efficient preconditioners for linear systems arising from Galerkin discretizations of *scalar* elliptic boundary value problems have become available [5, 16, 26, 32]. These techniques combined with conjugate gradient schemes achieve optimal multigrid complexity under minimal regularity assumptions. It is therefore natural to

*The work of this author is supported by the Deutsche Forschungsgemeinschaft.
†The work of this author is supported by the Graduiertenkolleg "Analyse und Konstruktion in der Mathematik" funded by the Deutsche Forschungsgemeinschaft at the RWTH Aachen.

explore the potential of such techniques for other problem classes. As a typical example we will focus here on the Stokes equations as a simplified model for the motion of an incompressible, viscous fluid in an n–dimensional domain $\Omega \subset I\!R^n$, where $n = 2$ or $n = 3$ are of primary interest. In the non–stationary case the velocity field $\vec{u} : \Omega \times [0, T] \to I\!R^n$ and the pressure $p : \Omega \to I\!R$ are well–known to be related by the system

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} - \nu \Delta \vec{u} + \text{ grad } p &= \vec{f} \quad \text{in } \Omega \times (0, T), \\ \text{div } \vec{u} &= 0 \quad \text{in } \Omega \times (0, T), \\ \vec{u}(x, 0) &= \vec{u}_0 \quad \text{in } \Omega, \\ \vec{u} &= 0 \quad \text{on } \partial\Omega \times [0, T], \\ \int_\Omega p(x, t) dx &= 0 \quad \text{for each } t \in (0, T). \end{aligned} \tag{1.1}$$

Here ν is the kinematic viscosity coefficient of the fluid (the inverse of the Reynolds number). Let

$$\vec{X} := H_0^1(\Omega)^n, \quad M = L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q(x) \; dx = 0 \right\}, \tag{1.2}$$

where $L^2(\Omega)$ and $H_0^1(\Omega)$ stand for the usual space of square integrable functions on Ω and the closure of the set of C^∞–functions with compact support in Ω relative to the norm $\|u\|_{H^1(\Omega)} := \left(\sum_{|\alpha| \le 1} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2}$, respectively. We will focus first on the stationary version of (1.1). A possible approach to solving (1.1) numerically can be based on the *Leray formulation* provided that *divergence free* trial functions are available (see e.g. [27]). However, here we choose the following alternative weak formulation. Find a pair $(\vec{u}, p) \in \vec{X} \times M$ such that

$$\begin{aligned} \nu \; a(\vec{u}, \vec{v}) + b(\vec{v}, p) &= \langle \vec{f}, \vec{v} \rangle \quad \text{for all } \vec{v} \in \vec{X}, \\ b(\vec{u}, \mu) &= 0 \quad \text{for all } \mu \in M, \end{aligned} \tag{1.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing for $\vec{X}$ and its dual $\vec{X}^*$, induced by the standard scalar product

$$(u, v)_{L^2(\Omega)} := \int_\Omega u(x) \; v(x) \; dx,$$

and the bilinear forms $a : \vec{X} \times \vec{X} \to I\!R$ and $b : \vec{X} \times M \to I\!R$ are defined by

$$\begin{aligned} a(\vec{u}, \vec{v}) &:= \sum_{i,j=1}^n \left(\frac{\partial}{\partial x_i} u_j, \frac{\partial}{\partial x_i} v_j \right)_{L^2(\Omega)} =: (\text{grad } \vec{u}, \; \text{grad } \vec{v})_{L^2(\Omega)^n}, \\ b(\vec{u}, q) &:= (\text{div } \vec{u}, q)_{L^2(\Omega)}, \end{aligned} \tag{1.4}$$

respectively.

In general, when the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous, it is well–known (see e.g. [3], p. 122) that (1.3) has a unique solution if and only if $a(\cdot,\cdot)$ is *elliptic* on the subspace

$$\vec{V} := \left\{ \vec{v} \in \vec{X} : \; b(\vec{v}, \mu) = 0, \;\; \mu \in M \right\},$$

i.e.

$$a(\vec{v}, \vec{v}) \geq \alpha \; \|\vec{v}\|^2_{\vec{X}} \quad \text{for all } \vec{v} \in \vec{V}, \tag{1.5}$$

and b satisfies the *inf–sup condition*

$$\inf_{\mu \in M} \sup_{\vec{v} \in \vec{X}} \frac{b(\vec{v}, \mu)}{\|\vec{v}\|_{\vec{X}} \; \|\mu\|_M} \geq \beta > 0. \tag{1.6}$$

In the particular case (1.4) the forms a and b are obviously continuous and a is even $\vec{X}$–elliptic where here $\vec{V} = \{\vec{v} \in \vec{X} : \; \operatorname{div} \vec{v} = 0\}$, so that it remains to verify (1.6). When Ω is a bounded simply connected domain with Lipschitz boundary, the inf–sup condition is known to hold for the situation at hand, see e.g. [3].

To solve (1.3) approximately one may choose finite dimensional trial spaces $\vec{X}_h \subset \vec{X}, M_h \subset M$. The classical Galerkin approach then requires finding $(\vec{u}_h, p_h) \in \vec{X}_h \times M_h$ satisfying

$$\begin{array}{lcll} \nu \; a(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) & = & \langle \vec{f}, \vec{v}_h \rangle & \text{for all } \vec{v}_h \in \vec{X}_h, \\ b(\vec{u}_h, \mu_h) & = & 0 & \text{for all } \mu_h \in M_h. \end{array} \tag{1.7}$$

The unique solvability of (1.7) for each mesh size h, its stable computability as well as estimates for the accuracy of the resulting solutions $\vec{u}_h, p_h$ are known to hinge on the validity of the *Ladyšenskaja–Babuška–Brezzi (LBB) condition*

$$\inf_{\mu_h \in M_h} \sup_{\vec{v}_h \in \vec{X}_h} \frac{b(\vec{v}_h, \mu_h)}{\|\vec{v}_h\|_{\vec{X}} \; \|\mu_h\|_M} \geq \tilde{\beta} > 0, \tag{1.8}$$

which is to hold uniformly in h.

Of course, given that (1.6) holds, (1.8) imposes conditions on the particular discretizations. The following result due to Fortin [19] (see also [3, 20]) offers a way to check the validity of (1.8).

Proposition 1.1 *Assume that the spaces $\vec{X}$ and M satisfy the inf–sup condition* (1.6). *Then condition* (1.8) *holds with some $\tilde{\beta} > 0$ uniformly in h if and only if there exist linear operators $Q_h : \vec{X} \to \vec{X}_h$ satisfying*

$$\|Q_h \vec{v}\|_{\vec{X}} \lesssim \|\vec{v}\|_{\vec{X}}, \quad \vec{v} \in \vec{X}, \tag{1.9}$$

and

$$b(\vec{v} - Q_h \vec{v}, \mu_h) = 0, \quad \vec{v} \in \vec{X}, \mu_h \in M_h. \tag{1.10}$$

Here $A \lesssim B$ means that A can be bounded by some constant multiple of B where the constant is independent of the various parameters the quantities A and B may depend on.

Several concrete examples of bivariate finite element spaces satisfying (1.8) are known. For three space dimensions the list of finite elements satisfying (1.8) is significantly shorter. In spite of the fact that in the finite element context the criterion in Proposition 1.1 is usually not very practicable, it turns out to be quite suitable in a somewhat different setting. In fact, utilizing Proposition 1.1, we propose in this paper a systematic construction of nested trial spaces $\vec{X}_h, M_h$ satisfying (1.8). We emphasize that the Stokes problem is to be viewed as one example and that the approach applies as well to related saddle point problems arising for instance from mixed formulations for elliptic problems.

Roughly speaking, the main idea can be described as follows. Instead of starting with a specific finite element space in $\vec{X}$, we construct a sequence of projectors $Q_{h_j} = Q_j$ satisfying (1.10) and take their ranges $\vec{X}_j = \vec{X}_{h_j}$ as trial spaces. The construction of the Q_j, in turn, is based on the construction of a suitable *multiscale basis*. In Section 2 we collect some relevant facts about such bases for later use. A convenient way for constructing these bases is provided by shift–invariant refinable spaces, often referred to as *multiresolution analyses*. This together with the construction of the projectors Q_j and corresponding multiscale bases adapted to the problem at hand is described in Section 3. It should be also emphasized that the approach works independently of the number of spatial variables. Moreover, the order of accuracy of the resulting trial spaces can easily be raised, at the expense of larger supports of the basis functions, of course.

There are a few more consequences which are worth mentioning. Firstly, stable multiscale bases offer a particularly convenient framework for local error control and adaptive techniques. This issue will be addressed in more detail elsewhere. The second issue concerns the numerical solution of the linear systems arising from (1.7). Defining $\mathbf{A}_j : \vec{X}_j \to \vec{X}_j, \mathbf{B}_j : \vec{X}_j \to M_j$ by

$$\nu\, a(\vec{u}_j, \vec{v}_j) = \langle \mathbf{A}_j \vec{u}_j, \vec{v}_j \rangle, \qquad b(\vec{u}_j, \mu_j) = (\mathbf{B}_j \vec{u}_j, \mu_j)_{L^2(\Omega)}, \tag{1.11}$$

it is clear that $\mathbf{A}_j$ is symmetric positive definite and that (1.7) amounts to solving the saddle point problem

$$\begin{pmatrix} \mathbf{A}_j & \mathbf{B}_j^* \\ \mathbf{B}_j & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{p}_j \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}. \tag{1.12}$$

The treatment of (1.12) is in one way or another tied to the Schur complement

$$\mathbf{K}_j := \mathbf{B}_j \mathbf{A}_j^{-1} \mathbf{B}_j^*$$

(see e.g. [4]). In case (1.4) and ν not too small $\mathbf{K}_j$ is well–conditioned. However, when employing implicit time stepping schemes for the non–stationary case (1.1), the condition number deteriorates with decreasing time steps. Building up on recent investigations in [4], we will point out in Section 4 that the multiscale bases not only give rise to stable discretizations in the sense of (1.8) but lead also to a convenient efficient preconditioner replacing the approach based on solving Neumann problems proposed in [4]. In Section 5 we construct a class of examples that fit the conditions required in Section 3. In Section 6 we comment on computational issues, in particular, pertaining to the computation of the entries of the right hand sides and stiffness matrices. This is another instance of taking essential advantage of the shift-invariance of the trial spaces. Refinability of the basis functions allows one to reduce these quadrature tasks to solving once a certain linear system whose size depends only on the supports of the generators of the multiresolution analysis but *not* on the level of discretization. We conclude with some concrete examples of stable pairs of trial spaces and present some numerical experiments for the *Driven Cavity* Problem for two and three space variables.

2 Multiscale Decompositions

In this section we collect a few general facts for later purposes. Suppose $\mathcal{S} = \{S_j\}_{j=0}^{\infty}$ is a sequence of closed nested subspaces of some Hilbert space H. Usually each space S_j is defined as linear span of some basis $\Phi_j = \{\varphi_{j,k} : k \in I_j\}$ which is to be *stable*, i.e.,

$$\|\mathbf{c}\|_{\ell^2(I_j)} \sim \Big\| \sum_{k\in I_j} c_k \; \varphi_{j,k} \Big\|_H, \tag{2.1}$$

where $A \sim B$ means that both relations $A \lesssim B$ and $B \lesssim A$ hold, and $\|\mathbf{c}\|_{\ell^2(I_j)} = \left(\sum_{k\in I_j} |c_k|^2\right)^{1/2}$. Φ_j typically consists of functions with compact support whose diameter remains uniformly proportional to the 'meshsize' h_j which in the following will be assumed for simplicity to behave like 2^{-j}.

To update a given coarse approximation $v_{j-1} \in S_{j-1}$ of some $v \in H$ it is convenient to decompose

$$S_j = S_{j-1} \oplus W_j$$

where W_j is some direct summand.

Suppose now that also a (uniformly) stable basis $\Psi_j = \{\psi_{j,k} : k \in J_j\}$ of each complement space W_j is known, i.e. Ψ_j satisfies (2.1) uniformly in j. Then any $v_m \in S_m$ can be written in single–scale representation as

$$v_m = \sum_{k\in I_m} c_k \; \varphi_{m,k},$$

or, with $\Psi_0 := \Phi_0,\ J_0 := I_0$, in multiscale form as

$$v_m = \sum_{j=0}^{m} \sum_{k\in J_j} d_{j,k} \; \psi_{j,k}.$$

The transformation

$$\mathbf{T}_m : \mathbf{d} \to \mathbf{c} \tag{2.2}$$

which takes the multiscale coefficients $d_{j,k}$ into the single scale coefficients c_k is of central importance in typical wavelet applications and will turn out to play a crucial role in the present context as well. It is therefore essential that the $\mathbf{T}_j$ are efficiently executable and well–conditioned. As for the first issue note that the nestedness of the spaces S_j implies the existence of refinement matrices $\mathbf{R}_{0,j} = (r^j_{l,k})_{l\in I_{j+1},k\in I_j}$ such that

$$\varphi_{j,k} = \sum_{l\in I_{j+1}} r^j_{l,k} \; \varphi_{j+1,l}, \quad k \in I_j. \tag{2.3}$$

In all typical applications the $\mathbf{R}_{0,j}$ are uniformly banded, i.e., all rows and columns of $\mathbf{R}_{0,j}$ contain only a uniformly bounded number of nonzero entries. It is clear that $\psi_{j,k}$ must have the form

$$\psi_{j+1,k} = \sum_{l \in I_{j+1}} r^j_{l,k} \; \varphi_{j+1,l}, \quad k \in J_{j+1}, \tag{2.4}$$

where the composite matrix $\mathbf{R}_j = (\mathbf{R}_{0,j}, \mathbf{R}_{1,j})$ with $\mathbf{R}_{1,j} = (r^j_{l,k})_{l \in I_{j+1}, k \in J_{j+1}}$, is invertible. One easily verifies (cf. [13]) that the transformation $\mathbf{T}_j$ has then the structure of a *pyramid* scheme similar to the fast wavelet transform

$$\mathbf{T}_j = \hat{\mathbf{R}}_1 \cdots \hat{\mathbf{R}}_j, \quad \hat{\mathbf{R}}_l = \begin{pmatrix} \mathbf{R}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{2.5}$$

As a consequence one has

Remark 2.1 *If* $\mathbf{R}_{0,j}, \mathbf{R}_{1,j}$ *(and thus* $\mathbf{R}_j$*) are uniformly banded, then the application of* $\mathbf{T}_j$ *requires only* $\mathcal{O}(\textit{dim}\ S_j)$ *operations.*

As for the condition number of $\mathbf{T}_j$, it is known that

$$\mathrm{cond}_2\,(\mathbf{T}_j) = \mathcal{O}(1) \tag{2.6}$$

if and only if $\mathbf{\Psi} = \{\psi_{j,k} : k \in J_j,\ j = 0, 1, \ldots\}$ is a Riesz basis for H [13]. This means that every $v \in H$ possesses a unique expansion

$$v = \sum_{j=0}^{\infty} \sum_{k \in J_j} d_{j,k}(v) \; \psi_{j,k}$$

and

$$\|v\|_H \sim \left(\sum_{j=0}^{\infty} \sum_{k \in J_j} |d_{j,k}(v)|^2 \right)^{1/2}. \tag{2.7}$$

With the aid of the Riesz–representation theorem one easily derives from (2.7) the existence of a dual Riesz basis $\tilde{\Psi} = \{\tilde{\psi}_{j,k} : k \in J_j,\ j = 0, 1, \ldots\}$, i.e.

$$(\psi_{j,k}, \tilde{\psi}_{j',k'})_H = \delta_{j,j'} \; \delta_{k,k'}, \quad k \in J_j,\ k' \in J_{j'},\ j, j' \in \mathbb{N}_0. \tag{2.8}$$

Thus the mappings

$$Q_j v := \sum_{l=0}^{j} \sum_{k \in J_l} (v, \tilde{\psi}_{l,k})_H \, \psi_{l,k}, \qquad Q^*_j v := \sum_{l=0}^{j} \sum_{k \in J_l} (v, \psi_{l,k})_H \, \tilde{\psi}_{l,k} \tag{2.9}$$

are uniformly bounded projectors with ranges S_j and $\tilde{S}_j$, respectively. Obviously, Q^*_j is the adjoint of Q_j. The following observations are useful (see e.g. [13]).

Remark 2.2 *Let* Q_j *be uniformly bounded linear projectors from some Hilbert space* H *onto nested closed subspaces* S_j *of* H*. Then the following properties are equivalent:*

(i) The Q_j commute, i.e.

$$Q_l Q_j = Q_l, \quad l \leq j, \tag{2.10}$$

(ii) $Q_j - Q_{j-1}$ are also projectors,

(iii) the ranges $\tilde{S}_j$ of the adjoints Q_j^* of Q_j are also nested.

Note that the particular projectors Q_j (2.9) induced by the Riesz basis do satisfy (2.10). Moreover, assuming that the Ψ_j are uniformly stable, then (2.7) is equivalent to

$$\|v\|_H \sim \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_H^2 \right)^{1/2}. \tag{2.11}$$

As for the validity of (2.11) and hence of (2.6) in such a general Hilbert space context we record some facts from [14]. To describe under which circumstances (2.11) holds, we call any subadditive uniformly bounded family of functionals $\omega(\cdot, t), t > 0$, satisfying $\lim_{t \to 0^+} \omega(v, t) = 0$ for $v \in H$, a *modulus*. $\mathcal{S}$ is said to satisfy a *Jackson* and *Bernstein* estimate relative to a modulus ω if there exists some $\gamma > 0$ such that the relations

$$\inf_{v_j \in S_j} \|v - v_j\|_H \lesssim \omega(v, 2^{-j}), \quad v \in H, \tag{2.12}$$

and

$$\omega(v_j, t) \lesssim \left(\min \{1, t2^j\} \right)^{\gamma} \|v_j\|_H, \quad v_j \in S_j, \tag{2.13}$$

hold uniformly in j, respectively.

Theorem 2.3 *For $\mathcal{S}, H$ as above, let Q_j be uniformly bounded linear projectors onto S_j satisfying* (2.10). *Assume that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ both satisfy Jackson and Bernstein estimates* (2.12), (2.13) *relative to some modulus ω for some $\gamma, \tilde{\gamma} > 0$, respectively. Then* (2.11) *holds.*

In view of the apparently pivotal role of condition (2.10), we record yet another equivalent formulation of Remark 2.2 (iii) for later use. The projectors Q_j can be represented as

$$Q_j v = \sum_{k \in I_j} (v, \tilde{\varphi}_{j,k})_H \; \varphi_{j,k}, \tag{2.14}$$

where the set $\tilde{\Phi}_j = \{\tilde{\varphi}_{j,k} : k \in I_j\}$ is biorthogonal to Φ_j, i. e.,

$$(\varphi_{j,k}, \tilde{\varphi}_{j,l})_H = \delta_{k,l}, \quad k, l \in I_j. \tag{2.15}$$

Remark 2.4 *The Q_j satisfy* (2.10) *if and only if $\tilde{\Phi}_j$ is also refinable. In fact, one has*

$$\tilde{\varphi}_{j,k} = \sum_{l \in I_{j+1}} \tilde{r}^j_{l,k} \; \tilde{\varphi}_{j+1,l}, \quad k \in I_j,$$

where $\tilde{r}^j_{l,k} = (\tilde{\varphi}_{j,k}, \varphi_{j+1,l})_H$.

We will apply Theorem 2.3 later to $H = L^2(\Omega)$ or $H = H^s$ for $s \in I\!R$, where H^s may stand for any of the spaces $H^s(\Omega), H^s_0(\Omega)$. (In fact, one could also assume homogeneous Dirichlet conditions on part of the boundary.) The role of the modulus is then played by the standard L^2-modulus of smoothness. Recall that the dth order L^2–modulus of smoothness is defined by

$$\omega_d(v,t)_{L^2(\Omega)} := \sup_{|h|\le t} \|\Delta_h^d v\|_{L^2(\Omega_{h,d})}, \tag{2.16}$$

where $\Omega_{h,d} := \{x \in \Omega : x + lh \in \Omega, l = 0, \dots, d\}$ and

$$\Delta_h^d v(\cdot) := \sum_{j=0}^{d} \binom{d}{j} (-1)^{d-j} v(\cdot + jh)$$

(see e.g. [16]). When the modulus $\omega(\cdot,t)$ in (2.13) is chosen as the dth order L^2-modulus defined in (2.16), it is well–known that, under mild assumptions on the regularity of the domain, the validity of (2.12), (2.13) is equivalent to the direct estimates

$$\inf_{v_j \in S_j} \|v - v_j\|_{L^2(\Omega)} \lesssim 2^{-sj}\, \|v\|_{H^s}, \quad v \in H^s,\ s \le d, \tag{2.17}$$

and the inverse estimates

$$\|v_j\|_{H^t} \lesssim 2^{j(t-s)}\, \|v_j\|_{H^s}, \quad v_j \in S_j,\ s \le t < \gamma, \tag{2.18}$$

where one usually has $\gamma \le d$. This yields the following norm equivalences (see e.g. [14]).

Proposition 2.5 *If the spaces S_j and $\tilde S_j$ satisfy* (2.17) *and* (2.18) *for some $d, \tilde d \in I\!N$ and $0 < \gamma \le d$, $0 < \tilde\gamma \le \tilde d$, then one has*

$$\|v\|_{H^t} \sim \left(\sum_{j=0}^{\infty} 2^{2tj} \left\| (Q_j - Q_{j-1})v \right\|^2_{L^2(\Omega)} \right)^{1/2}, \quad t \in (-\tilde\gamma, \gamma), \tag{2.19}$$

and

$$\|v\|_{H^t} \sim \left(\sum_{j=0}^{\infty} \left\| (Q_j - Q_{j-1})v \right\|^2_{H^t} \right)^{1/2}, \quad t \in (-\tilde\gamma, \gamma). \tag{2.20}$$

A useful interpretation of these facts may be formulated as follows. Let

$$\Lambda_s v := \sum_{j=0}^{\infty} 2^{js}\, (Q_j - Q_{j-1})v. \tag{2.21}$$

Then, under the assumptions in Proposition 2.5,

$$\|\Lambda_s v\|_{H^t} \sim \|v\|_{H^{t+s}} \ \text{ for } t + s \in (\tilde\gamma, \gamma). \tag{2.22}$$

This latter fact will play an important role for the issue of preconditioning.

We conclude this section with a comment on the condition (2.17) which is taylored to uniform mesh refinements. Equivalences of the form (2.19) actually persist to hold under weaker assumptions permitting spaces, S_j resulting from adaptive refinements. In fact, one way to show that (2.19) still holds, is to establish an estimate of the form

$$\|(Q_j - Q_{j-1})v_m\|_{L^2(\Omega)} \lesssim \omega_d(v_m, 2^{-j})_{L^2(\Omega)}, \quad v_m \in S_m,\ j \le m,$$

see [16].

3 Multiscale Bases for the Stokes Problem

Combining the facts stated in the previous section with Proposition 1.1 suggests constructing multiscale basis functions $\psi_{j,k}$ in such a way that the corresponding projectors Q_j, defined by (2.9), satisfy on one hand (1.10) and on the other hand (2.10) as well as relations of the type (2.17), (2.18) to ensure norm equivalences of the form (2.19). First we will show that this task can conveniently be solved when $\Omega = I\!R^n$. In a second step we will indicate possible ways of adapting the construction to bounded domains.

3.1 The Shift–invariant Case and Wavelets

The main ingredient of our construction is the concept of *biorthogonal wavelets*. Recall from Remark 2.4 that in the above general context biorthogonality was expressed through condition (2.10). Suppose that $\xi, \tilde{\xi}$ are compactly supported functions in $L^2(I\!R)$ which are *refinable*, i.e., the relations

$$\xi(x) = \sum_{k \in \mathbb{Z}} a_k \, \xi(2x-k), \quad \tilde{\xi}(x) = \sum_{k \in \mathbb{Z}} \tilde{a}_k \, \tilde{\xi}(2x-k), \tag{3.1.1}$$

hold for some *masks* $\mathbf{a} = \{a_k\}_{k \in \mathbb{Z}}$, $\tilde{\mathbf{a}} = \{\tilde{a}_k\}_{k \in \mathbb{Z}}$. We will call $\xi, \tilde{\xi}$ a *dual pair* if one has in addition

$$\left(\xi, \tilde{\xi}(\cdot - k)\right)_{L^2(I\!R^n)} = \delta_{0,k}, \quad k \in \mathbb{Z}. \tag{3.1.2}$$

It is easy to see that (3.1.2) forces both masks to be finitely supported when ξ and $\tilde{\xi}$ have compact support. Moreover, defining for $g \in L^2(I\!R)$

$$g_{j,k} := 2^{j/2} g(2^j \cdot -k), \quad j,k \in \mathbb{Z},$$

the functions derived from ξ and $\tilde{\xi}$ by dilation and integer shifts are *stable* in the sense of

$$\|\mathbf{c}\|_{\ell^2(\mathbb{Z})} \sim \left\| \sum_{k \in \mathbb{Z}} c_k \, \xi_{j,k} \right\|_{L^2(I\!R)}. \tag{3.1.3}$$

Examples of such dual pairs can be found in [11].

Denoting in the following for any collection F of functions in L^2 by $S(F)$ the L^2-closure of the span of F and setting

$$(\xi)_j := \{\xi_{j,k} : k \in \mathbb{Z}\},$$

(3.1.1) implies that the spaces $S_j := S((\xi)_j), \tilde{S}_j := S((\tilde{\xi})_j)$ are nested, i.e.,

$$S_j \subset S_{j+1}, \quad \tilde{S}_j \subset \tilde{S}_{j+1}, \quad j \in \mathbb{Z}. \tag{3.1.4}$$

This obviously fits into the framework described in the previous section. Moreover, it is easy to identify suitable complement bases. In fact, it is known that ([9, 11])

$$\eta(x) = \sum_{k \in \mathbb{Z}} (-1)^k \, \tilde{a}_{1-k} \, \xi(2x-k), \quad \tilde{\eta}(x) = \sum_{k \in \mathbb{Z}} (-1)^k \, a_{1-k} \, \tilde{\xi}(2x-k), \tag{3.1.5}$$

satisfy

$$(\eta, \tilde{\eta}(\cdot - k))_{L^2(\mathbb{R})} = \delta_{0,k}, \quad (\xi, \tilde{\eta}(\cdot - k))_{L^2(\mathbb{R})} = (\tilde{\xi}, \eta(\cdot - k))_{L^2(\mathbb{R})} = 0, \quad k \in \mathbb{Z}. \tag{3.1.6}$$

One readily concludes from (3.1.1) and (3.1.6) that

$$(\eta_{j,k}, \tilde{\eta}_{j',k'})_{L^2(\mathbb{R}^n)} = \delta_{(j,k),(j',k')}, \quad j, j', k, k' \in \mathbb{Z}, \tag{3.1.7}$$

(see (2.8)). Thus the $\eta_{j,k}$ give rise to a stable multiscale basis in the sense of Section 2. Moreover, the complement spaces $W_j := S((\eta)_j), \tilde{W}_j := S((\tilde{\eta})_j)$ satisfy

$$\tilde{W}_j \perp S_{j-1}, \quad W_j \perp \tilde{S}_{j-1}, \quad W_j \perp \tilde{W}_{j'}, \quad j \neq j'. \tag{3.1.8}$$

3.2 Modifying Dual Pairs

One reason for considering the above shift–invariant setting is that, given a dual pair of generators ξ, $\tilde{\xi}$, there is a relatively simple mechanism of generating new dual pairs ξ^*, $\tilde{\xi}^*$ which will turn out to be useful for constructing multiscale bases adapted to the Stokes problem. To describe this it is convenient to introduce for a given mask $\mathbf{a}$ of refinement coefficients appearing in (3.1.1) its *symbol*

$$\mathbf{a}(z) := \sum_{k \in \mathbb{Z}} a_k \, z^k, \qquad z \in \mathbb{C}. \tag{3.2.1}$$

In fact, the biorthogonality relation (3.1.2) implies

$$\mathbf{a}(z)\overline{\tilde{\mathbf{a}}(z)} + \mathbf{a}(-z)\overline{\tilde{\mathbf{a}}(-z)} = 4. \tag{3.2.2}$$

Moreover, when $\mathbf{a}(z)$ is divisible by $(1+z)$ (which is known to be the case when $\xi \in H^1(\mathbb{R})$) it is clear that the new pair of symbols

$$\mathbf{b}(z) := \frac{2}{1+z}\mathbf{a}(z), \quad \tilde{\mathbf{b}}(z) = \frac{1+\overline{z}}{2}\tilde{\mathbf{a}}(z),$$

still satisfies (3.2.2). Thus if the new masks $\mathbf{b}, \tilde{\mathbf{b}}$ still admit solutions to (3.1.1) one ends up with a new dual pair which turns out to be related to the initial one through differentiation and integration. This is made precise by the following observation which is essentially due to Lemarié–Rieusset [24, 25]. It is a special case of a result established in [27].

Lemma 3.1 *Let ξ, $\tilde{\xi} \in L^2(\mathbb{R})$ be compactly supported dual functions, which are refinable with masks $\mathbf{a}^0$ and $\tilde{\mathbf{a}}^0$, respectively. If $\xi \in H^1(\mathbb{R})$ and $\int_{\mathbb{R}} \xi(x)\,dx = \int_{\mathbb{R}} \tilde{\xi}(x)\,dx = 1$, then there exists a dual pair ξ^*, $\tilde{\xi}^*$ of compactly supported functions in $L^2(\mathbb{R})$ such that*

$$\frac{d}{dx}\xi(x) = \xi^*(x) - \xi^*(x-1) \quad \text{and} \quad \frac{d}{dx}\tilde{\xi}^*(x) = \tilde{\xi}(x+1) - \tilde{\xi}(x) \tag{3.2.3}$$

holds. Moreover, their symbols satisfy the relations

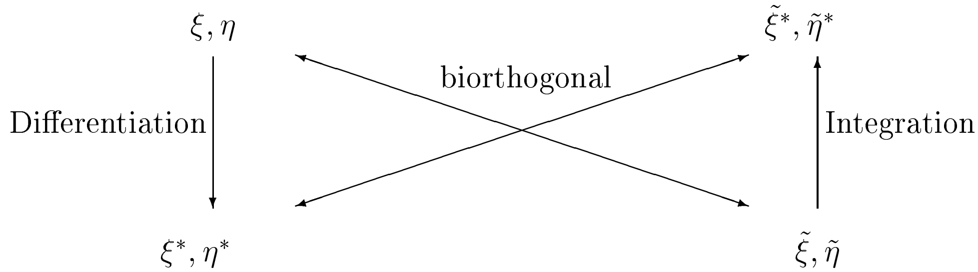
$$\mathbf{a}^{(0,*)}(z) = \frac{2}{1+z}\mathbf{a}^0(z), \quad \tilde{\mathbf{a}}^{(0,*)}(z) = \frac{1+\bar{z}}{2}\tilde{\mathbf{a}}^0(z), \tag{3.2.4}$$

and the relations

$$\frac{d}{dx}\eta(x) = 4\ \eta^*(x), \quad \text{and} \quad \frac{d}{dx}\tilde{\eta}^*(x) = -4\ \tilde{\eta}(x) \tag{3.2.5}$$

are valid for the biorthogonal wavelets defined by (3.1.5).

Note that since, by (3.2.3), $\tilde{\xi}^*$ belongs to $H^1(\mathbb{R})$, $\tilde{\xi}^*$ must be continuous. The essence of the above statements is that, starting with a refinable dual pair $\xi, \tilde{\xi}$, a certain modification produces a new refinable dual pair $\xi^*, \tilde{\xi}^*$, thus preserving the crucial property (2.10). The new pair will serve as the main building block for the construction of pairs of trial spaces satisfying the LBB condition. The advantage of the shift–invariant setting lies in the fact that the whole construction reduces to manipulating Laurent polynomials. The modification consists essentially of differentiation and integration illustrated by the following figure:



The most convenient way of constructing multivariate dual pairs of refinable functions is to employ tensor products, i.e., for $x \in \mathbb{R}^n$ and any univariate dual pair of *univariate* refinable functions $\xi, \tilde{\xi}$

$$\phi(x) := \xi(x_1)\cdots\xi(x_n), \quad \tilde{\phi}(x) := \tilde{\xi}(x_1)\cdots\tilde{\xi}(x_n), \tag{3.2.6}$$

are easily seen to form a multivariate dual pair. More generally, writing

$$\xi_0 := \xi, \quad \tilde{\xi}_0 := \tilde{\xi}, \quad \xi_1 := \eta, \quad \tilde{\xi}_1 := \tilde{\eta},$$

and analogously for $\xi_e^*, \tilde{\xi}_e^*$, $e \in \{0,1\}$, the corresponding multivariate scaling functions and wavelets are given by

$$\psi_e(x) := \prod_{i=1}^{n} \xi_{e_i}(x_i), \quad \tilde{\psi}_e(x) := \prod_{i=1}^{n} \tilde{\xi}_{e_i}(x_i), \quad e \in \{0,1\}^n =: E, \tag{3.2.7}$$

where, of course, $\psi_0 = \phi, \tilde{\psi}_0 = \tilde{\phi}$. Moreover, defining the modified functions $\psi_e^{(\nu)}, \tilde{\psi}_e^{(\nu)}$, $e \in E$, by

$$\psi_e^{(\nu)}(x) = \xi_{e_1}(x_1)\cdots\xi_{e_{\nu-1}}(x_{\nu-1})\ \xi_{e_\nu}^*(x_\nu)\ \xi_{e_{\nu+1}}(x_{\nu+1})\cdots\xi_{e_n}(x_n), \tag{3.2.8}$$

as well as

$$\tilde{\psi}_e^{(\nu)}(x) = \tilde{\xi}_{e_1}(x_1)\cdots\tilde{\xi}_{e_{\nu-1}}(x_{\nu-1})\ \tilde{\xi}_{e_\nu}^*(x_\nu)\ \tilde{\xi}_{e_{\nu+1}}(x_{\nu+1})\cdots\tilde{\xi}_{e_n}(x_n), \tag{3.2.9}$$

Lemma 3.1 yields

$$\frac{\partial}{\partial x_\nu}\psi_e(x) = \begin{cases} \nabla_\nu \psi_e^{(\nu)}(x), & \text{if } e_\nu = 0, \\ 4\ \psi_e^{(\nu)}(x), & \text{if } e_\nu = 1, \end{cases} \tag{3.2.10}$$

and

$$\frac{\partial}{\partial x_\nu}\tilde{\psi}_e^{(\nu)}(x) = \begin{cases} \Delta_\nu \tilde{\psi}_e(x), & \text{if } e_\nu = 0, \\ (-4)\ \tilde{\psi}_e(x), & \text{if } e_\nu = 1, \end{cases} \tag{3.2.11}$$

where

$$\nabla_\nu f := f(\cdot) - f(\cdot - e^\nu), \quad \Delta_\nu f := f(\cdot + e^\nu) - f(\cdot),$$

and the e^ν are the coordinate vectors, i.e., $(e^\nu)_{\nu'} = \delta_{\nu,\nu'}, \nu, \nu' = 1, \ldots, n$. Again we have used the convention $\tilde{\psi}_0^{(\nu)} = \tilde{\phi}^{(\nu)}$ and $\psi_0^{(\nu)} = \phi^{(\nu)}$.

To indicate the relevance of the above manipulations with regard to the LBB condition, let for $E = \{0,1\}^n$, $E^* := E \setminus \{0\}$

$$\psi_{j,k} := 2^{nj/2}\psi_e(2^j \cdot - \alpha), \quad \tilde{\psi}_{j,k} := 2^{nj/2}\tilde{\psi}_e(2^j \cdot - \alpha), \quad k = (e,\alpha) \in E \times \mathbb{Z}^n, \tag{3.2.12}$$

and

$$\begin{aligned} \Phi_j &:= \{\phi_{j,k} : k \in \{0\} \times \mathbb{Z}^n\}, \quad \tilde{\Phi}_j = \{\tilde{\phi}_{j,k} : k \in \{0\} \times \mathbb{Z}^n\}, \\ \Psi_j &:= \{\psi_{j,k} : k \in E^* \times \mathbb{Z}^n\}, \quad \tilde{\Psi}_j = \{\tilde{\psi}_{j,k} : k \in E^* \times \mathbb{Z}^n\}. \end{aligned} \tag{3.2.13}$$

One easily derives from the biorthogonality of the univariate factors (3.1.7) that

$$\left(\psi_{j,k}, \tilde{\psi}_{j',k'}\right)_{L^2(\mathbb{R}^n)} = \delta_{j,j'}\ \delta_{k,k'}, \quad j, j' \in \mathbb{N},\ k, k' \in E^* \times \mathbb{Z}^n. \tag{3.2.14}$$

In other words, this means

$$S(\tilde{\Psi}_j) \perp S(\Phi_{j-1}), \quad S(\Psi_j) \perp S(\tilde{\Phi}_{j-1}), \quad S(\Psi_j) \perp S(\tilde{\Psi}_{j'}) \text{ for } j \neq j'. \tag{3.2.15}$$

Analogous relations hold for $\Phi_j, \tilde{\Phi}_j$ replaced by $\Phi_j^{(\nu)}, \tilde{\Phi}_j^{(\nu)}$, $\nu = 1, \ldots, n$. Now set

$$M_j := S(\Phi_j), \quad \vec{X}_j := S(\tilde{\Phi}_j^{(1)}) \times \cdots \times S(\tilde{\Phi}_j^{(n)}), \tag{3.2.16}$$

and define in analogy to (2.14) the projectors $\vec{Q}_j$ by

$$(\vec{Q}_j \vec{v})_\nu := \sum_{k \in \{0\} \times \mathbb{Z}^n} \left(\vec{v}_\nu, \phi_{j,k}^{(\nu)}\right)_{L^2(\mathbb{R}^n)} \tilde{\phi}_{j,k}^{(\nu)}, \quad \nu = 1, \ldots, n.$$

A glance at the corresponding multiscale representation (2.9) reveals that for any $\vec{v} \in \vec{X}$

$$\vec{v}_\nu - (\vec{Q}_m \vec{v})_\nu = \sum_{j=m+1}^{\infty} \sum_{k \in E^* \times \mathbb{Z}^n} (\vec{v}_\nu, \psi_{j,k}^{(\nu)})_{L^2(\mathbb{R}^n)}\ \tilde{\psi}_{j,k}^{(\nu)} =: \sum_{j=m+1}^{\infty} (\vec{w}_j)_\nu.$$

Obviously, $\vec{w}_j \in S(\tilde{\Psi}_j^{(1)}) \times \cdots \times S(\tilde{\Psi}_j^{(n)})$. We readily obtain from (3.2.11) that

$$\operatorname{div} \vec{w}_j = \sum_{k \in E^* \times \mathbb{Z}^n} \tilde{d}_k\ \tilde{\psi}_{j,k}$$

for some coefficients $\tilde{d}_k$ and, hence, $\operatorname{div}\vec{w}_j$ belongs to the complement $S(\tilde{\Psi}_j)$ of $S(\tilde{\Phi}_{j-1})$ in $S(\tilde{\Phi}_j)$. But by (3.2.15), we have

$$S(\tilde{\Psi}_j) \perp S(\Phi_{j-1})$$

which means here that

$$\left(\operatorname{div}\vec{w}_j, v_m\right)_{L^2(\mathbb{R}^n)} \;=\; b(\vec{w}_j, v_m) \;=\; 0, \quad v_m \in S(\Phi_m) = M_m,\; j > m.$$

Hence the projectors $\vec{Q}_j$ satisfy the conditions (1.9) and (1.10) in Proposition 1.1 which shows that the pairs $M_j, \vec{X}_j$ satisfy the LBB condition. Thus the LBB condition on all of $\mathbb{R}^n$ amounts to the construction of suitably interrelated biorthogonal multiscale bases. Our central objective is now to extend this mechanism to bounded domains. Defining for any collection V of functions $\partial_\nu V := \{\frac{\partial}{\partial x_\nu} v : v \in V\}$, it will be useful to keep in mind that the essence of the above argument is the relation

$$S(\partial_\nu \tilde{\Psi}_j^{(\nu)}) \;\subseteq\; S(\tilde{\Psi}_j), \quad \nu = 1, \ldots, n. \tag{3.2.17}$$

3.3 Bounded Domains

It remains to establish multiscale bases for a given bounded domain. One possible strategy is to work with restrictions of the spaces introduced in the last section to a given domain and enforce essential boundary conditions by means of Lagrange multipliers. For scalar elliptic problems this approach is studied in [22]. The advantage would be to preserve possibly many properties of the shift-invariant multiresolution spaces while still being able to treat relatively general domain geometries. On the other hand, the saddle point problems become more complicated. These issues will be addressed in a forthcoming paper. An alternative is to adapt the multiscale bases to the given domain by suitable modifications of basis functions near the boundary.

The central task will then be to preserve the relation (3.2.17) under these modifications. A general construction of multiresolution spaces and their stable decompositions for essentially Lipschitz domains is proposed and analyzed in [8]. To focus on the essential ideas and to avoid the technicalities entailed by the general case we will confine the subsequent discussions to the simple model case $\Omega = [0, 1]^n$. In fact, the type of basis functions will be the same in the general case where, however, the boundary near modifications depend on the local behavior of the boundary. Since our main concern here is the LBB condition we have decided not to address the general boundary adaptation here.

Throughout the rest of this paper let $\Omega = [0, 1]^n$. Employing as above tensor products of univariate functions (3.2.6) then reduces the problem to constructing suitable multiscale bases on $[0, 1]$. Several such constructions have been described in the literature [1, 7, 10, 12, 21]. In particular, [1] treats the case of biorthogonal wavelets which is needed here. Unfortunately, one cannot apply these results directly since, on one hand, the stability in the sense of (2.11) is not addressed there and, on the other hand, the stability properties needed here require estimates of the form (2.17), (2.18) for the spaces spanned by both the generator and its dual, which is not given in any of these papers. When indicating the corresponding necessary modifications we also collect the requirements on the generators which will be relevant for subsequent applications and which will direct us later when giving concrete examples.

In view of (3.2.6), we will again be concerned with a dual pair $\xi, \tilde{\xi}$ of univariate refinable functions. It will be seen that all relevant properties of the resulting spaces can be conveniently expressed in terms of the *regularity* and *exactness* of $\xi, \tilde{\xi}$ in the following sense. We will assume that $\xi, \tilde{\xi}$ are supported in $[-l, l]$ for some fixed $l \in \mathbb{N}$. Firstly, we require that

$$\xi, \tilde{\xi} \in H^t(\mathbb{R}) \quad \text{for some} \quad t > 1. \tag{3.3.1}$$

In fact, for the stationary problem it would be sufficient if only one of the generators has Sobolev regularity exceeding one. Secondly, $\xi, \tilde{\xi}$ are to be exact of degree $d-1, \tilde{d}-1$, for some $d, \tilde{d} \in \mathbb{N}$, respectively, where, in particular, we will always assume that

$$\tilde{d} \geq d \geq 2. \tag{3.3.2}$$

Here ξ to be exact of degree $d-1$ means that for $r = 0, \ldots, d-1$

$$x^r = \sum_{k \in \mathbb{Z}} \Big((\cdot)^r, \tilde{\xi}(\cdot - k) \Big)_{L^2(\mathbb{R})} \xi(x-k). \tag{3.3.3}$$

It is well–known that the refinability by itself already implies that (3.3.3) holds for $r = 0$ [6]. Many examples of dual pairs with higher degree of exactness can be found in the literature. If one allows l to become large the parameters $t, d, \tilde{d}$ can be made arbitrarily large [11]. It is also well-known that when (3.3.3) holds, linear combinations of the dilates $\xi(2^j \cdot -k), k \in \mathbb{Z}$, provide approximation orders $\mathcal{O}(2^{-jd}), j \to \infty$, for functions in $H^d(\mathbb{R})$.

Remark 3.2 *Note that, since the exactness of a refinable function is known to be determined by the power of the factor $(1+z)$ in the symbol of its mask, the modified function $\tilde{\xi}^*$ given in Lemma 3.1 is under the above assumptions exact of degree $\tilde{d}$.*

We recall first the basic principle of constructing multiresolution spaces on $[0,1]$ generated by any given dual pair $\xi, \tilde{\xi}$. The key idea is to retain as much structure as possible from the spaces defined on all of $\mathbb{R}$ while preserving the degree of exactness. It is clear that when working just with the restrictions of the shifts $\xi(2^j \cdot -k)$ to $[0,1]$ one would loose biorthogonality and stability. Thus one keeps possibly many shifts $\xi(2^j \cdot -k)$ whose support is strictly inside $[0,1]$ as basis functions while modifying those interfering with the end points of the interval in such a way that overall one obtains stable biorthogonal basis functions which are still refinable and span all polynomials up to a desired degree. To be precise, fix integers $N, M > l, d$, set

$$K_j := K_{j,L} \cup K_{j,I} \cup K_{j,R}, \tag{3.3.4}$$

where

$$K_{j,L} := \{N-d, \ldots, N-1\}, \;\; K_{j,I} := \{N, \ldots, 2^j - M\}, \;\; K_{j,R} := \{2^j - M + 1, \ldots, 2^j - M + d\}$$

for all $j \geq j_0$ with j_0 big enough so that $K_{j,L}$ and $K_{j,R}$ do not overlap. In principle, one could take $N = M$. The reason for introducing a further parameter will become clear later. Now let

$$\xi_{j,k} := 2^{j/2}\xi(2^j \cdot -k), \qquad \tilde{\xi}_{j,k} := 2^{j/2}\tilde{\xi}(2^j \cdot -k), \quad k \in K_{j,I}. \tag{3.3.5}$$

Furthermore, in view of (3.3.3), define for $r = 0, \ldots, d-1$

$$\xi^L_{j,N-d+r} := 2^{j/2} \sum_{m=-l}^{N-1} 2^j \left((2^j \cdot)^r, \tilde{\xi}(2^j \cdot -m)\right)_{L^2(\mathbb{R})} \xi(2^j \cdot -m) \mid_{[0,1]} \tag{3.3.6}$$

and

$$\xi^R_{j,2^j-M+d-r} := 2^{j/2} \sum_{m=2^j-M+1}^{2^j+l} 2^j \left((2^j(1-\cdot))^r, \tilde{\xi}(2^j \cdot -m)\right)_{L^2(\mathbb{R})} \xi(2^j \cdot -m) \mid_{[0,1]}. \tag{3.3.7}$$

On account of (3.3.3), it is clear that the functions $\xi_{j,k}, k \in K_{j,I}$, together with the functions $\xi^L_{j,k}, k \in K_{j,L}$, $\xi^R_{j,k}, k \in K_{j,R}$, span all polynomials of degree at most $d-1$ on $[0,1]$. The functions $\tilde{\xi}^L_{j,k}, \tilde{\xi}^R_{j,k}$ are defined analogously with the roles of ξ and $\tilde{\xi}$ interchanged. For simplicity, we confine ourselves here to reproduce thereby the same degree $d-1$ of exactness also for the dual side (see [17] for the general case). As in [1] one can verify that the $\xi^L_{j,k}, \xi^R_{j,k}$ are linearly independent and refinable. It remains to biorthogonalize the sets $\xi^L_{j,k}, \tilde{\xi}^L_{j,k}, k \in K_{j,L}$, and $\xi^R_{j,k}, \tilde{\xi}^R_{j,k}, k \in K_{j,R}$, respectively. For instance, it will be convenient for later purposes to set

$$\tilde{\xi}_{j,k} := \tilde{\xi}^L_{j,k}, \quad k \in K_{j,L}, \qquad \tilde{\xi}_{j,k} := \tilde{\xi}^R_{j,k}, \quad k \in K_{j,R}. \tag{3.3.8}$$

One then has to determine coefficients $L_{j,k,m}, R_{j,k,m}$ such that the functions

$$\begin{aligned} \xi_{j,k} &:= \sum_{m \in K_{j,L}} L_{j,k,m}\, \xi^L_{j,m}, \quad k \in K_{j,L}, \\ \xi_{j,k} &:= \sum_{m \in K_{j,R}} R_{j,k,m}\, \xi^R_{j,m}, \quad k \in K_{j,R}, \end{aligned} \tag{3.3.9}$$

satisfy

$$\left(\xi_{j,k}, \tilde{\xi}_{j,m}\right)_{L^2([0,1])} = \delta_{k,m}, \quad k,m \in K_{j,L} \cup K_{j,R}. \tag{3.3.10}$$

Defining now

$$\Xi_j := \{\xi_{j,k} : k \in K_j\}, \quad \tilde{\Xi}_j := \{\tilde{\xi}_{j,k} : k \in K_j\}, \tag{3.3.11}$$

the following facts can be verified (see also [1, 17]).

Proposition 3.3 *Under the above assumptions one has:*

(i) diam supp $(\xi_{j,k}) \sim 2^{-j}$, $k \in K_j$, $\#K_j \sim 2^j$.

(ii) The Ξ_j *are uniformly stable.*

(iii) $\Pi_r([0,1]) \subset S(\Xi_j),\ r = 0,\dots,d-1,\ j \in \mathbb{N},\ j \ge j_0$, *where* $\Pi_r([0,1])$ *denotes the space of all polynomials of degree* r *on* $[0,1]$.

(iv) $S(\Xi_j) \subset S(\Xi_{j+1}),\ j \in \mathbb{N},\ j \ge j_0$.

Analogous facts hold for $\tilde{\Xi}_j$.

Note next that the quantities

$$2^{j/2}\left((2^j\cdot)^r, 2^{j/2}\tilde{\xi}(2^j\cdot -m)\right)_{L^2(\mathbb{R}^n)} = \left((\cdot)^r, \tilde{\xi}(\cdot - m)\right)_{L^2(\mathbb{R}^n)} =: \alpha^L_{r,m} \tag{3.3.12}$$

are actually independent of the level j. Defining $\alpha^R_{r,m}, \tilde{\alpha}^L_{r,m}, \tilde{\alpha}^R_{r,m}$ in an analogous fashion by exchanging $(\cdot)^r$ by $(1-\cdot)^r$ and $\tilde{\xi}$ by ξ, respectively, the relations (3.3.6), (3.3.7) take the form (with $N = M$)

$$\begin{aligned} \xi^L_{j,N-d+r} &= \sum_{m=-l}^{N-1} \alpha^L_{r,m} 2^{j/2}\xi(2^j\cdot -m)\mid_{[0,1]}, \\ \xi^R_{j,2^j-N+d-r} &= \sum_{m=2^j-N+1}^{2^j+l} \alpha^R_{r,m} 2^{j/2}\xi(2^j\cdot -m)\mid_{[0,1]}, \quad r = 0,\dots d-1, \end{aligned} \tag{3.3.13}$$

and analogously $\tilde{\xi}^L_{j,N-d+r}, \tilde{\xi}^R_{j,2^j-N+d-r}$. Moreover, it is well–known that the members $\xi, \tilde{\xi}$ of any dual pair can be normalized so that $\int_{\mathbb{R}} \xi(x)dx = \int_{\mathbb{R}} \tilde{\xi}(x)dx = 1$ so that, in particular,

$$\alpha^G_{0,m} = \tilde{\alpha}^G_{0,m} = 1, \quad G \in \{L, R\},\ m \in \mathbb{Z}. \tag{3.3.14}$$

Likewise it is easy to see that the coefficients $L_{j,k,m}, R_{j,k,m}$ in (3.3.9) do not depend on j and hence have to be computed only once [17].

It remains to determine the corresponding biorthogonal wavelets adapted to the interval. Again one retains possibly many wavelets in the form (3.1.5) as long as their support lies strictly in $(0,1)$. For the construction of the modifications near the endpoints we refer to [1, 12, 17]. Moreover, the explicit form of the modified scaling functions and wavelets for the present particular situation are recorded in [17]. Here it is important to note that these coefficients are also independent of the level j and therefore have to be determined only once by solving small linear systems.

Remark 3.4 *The particular choice of the boundary functions makes it easy to incorporate homogeneous boundary conditions. The case of interest is*

$$\tilde{\Xi}_{j,0} := \tilde{\Xi}_j \setminus \{\tilde{\xi}_{j,N-d}, \tilde{\xi}_{j,2^j-N+d}\}. \tag{3.3.15}$$

Note that, by construction, one can still represent near the boundary all polynomials of degree $d-1$ *which vanish at the boundary as linear combinations of the elements in* $\tilde{\Xi}_{j,0}$.

It will be useful to keep the following observation in mind.

Remark 3.5 *Independently of the choice of* $\xi, \tilde{\xi}, N, M$ *one has*

$$\dim\,(S(\Xi_j)) - \dim\,(S(\Xi_{j-1})) = \#K_j - \#K_{j-1} = \#K_{j,I} - \#K_{j-1,I} = 2^{j-1}.$$

Suitable spaces on $\Omega = [0,1]^n$ are now again obtained by taking tensor products. For $I_j := K_j \times \cdots \times K_j$ define

$$\Phi_j := \bigotimes_{i=1}^{n} \Xi_j = \{\phi_{j,k} := \xi_{j,k_1} \otimes \cdots \otimes \xi_{j,k_n} \;:\; k = (k_1, \ldots, k_n) \in I_j\}, \tag{3.3.16}$$

and analogously $\tilde{\Phi}_j$, where for $x = (x_1, \ldots x_n)^T$ we set $(v_1 \otimes \cdots \otimes v_n)(x) := v_1(x_1) \cdots v_n(x_n)$.

3.4 The Ladyšenskaja–Babuška–Brezzi Condition

In the sequel we will always assume that (3.3.1) holds, and that the bases $\Xi_j, \tilde{\Xi}_j$ are defined by (3.2.4) where $N = M$ in the definition of K_j (3.3.4). Given the bases $\Phi_j, \tilde{\Phi}_j$ on Ω, defined by (3.3.16) relative to the multivariate dual pair $\phi, \tilde{\phi}$ of the form (3.2.6), our main objective is to construct next bases $\Phi_j^{(\nu)}, \tilde{\Phi}_j^{(\nu)}$, $\nu = 1, \ldots n$, according to the modifications from Lemma 3.1. Since we have assumed that ξ, ξ are supported in $[-l, l]$ it follows from (3.2.3) that

$$\text{supp } \xi^* \subseteq [-l, l-1], \quad \text{supp } \tilde{\xi}^* \subseteq [-1-l, l]. \tag{3.4.1}$$

Moreover, since ξ^* essentially results from differentiation its degree of exactness is expected to drop by one. This suggests the following partition for the corresponding index sets K_j^* to be properly related to K_j (see also [29]).

$$K_j^* := K_{j,L}^* \;\cup\; K_{j,I}^* \;\cup\; K_{j,R}^*,$$

where

$$\begin{aligned} K_{j,L}^* &:= \{N-d+1, \ldots, N-1\}, \quad K_{j,I}^* := \{N, \ldots, 2^j - N + 1\}, \\ K_{j,R}^* &:= \{2^j - N + 2, \ldots, 2^j - N + d\}, \end{aligned} \tag{3.4.2}$$

i.e., here we have $M = N - 1$. The collections $\Xi_j^* = \{\xi_{j,k}^* : k \in K_j^*\}$ are the defined as described in Section 3.3. Similarly we set

$$\tilde{K}_j^* := \tilde{K}_{j,L}^* \;\cup\; \tilde{K}_{j,I}^* \;\cup\; \tilde{K}_{j,R}^*,$$

where

$$\begin{aligned} \tilde{K}_{j,L}^* &:= \{N-d, \ldots, N\}, \quad \tilde{K}_{j,R}^* := \{2^j - N + 1, \ldots, 2^j - N + d + 1\}, \\ \tilde{K}_{j,I}^* &:= \{N+1, \ldots, 2^j - N\}, \\ \tilde{K}_{j,L,0}^* &:= \{N-d+1, \ldots, N\}, \quad \tilde{K}_{j,R,0}^* := \{2^j - N + 1, \ldots, 2^j - N + d\}, \end{aligned} \tag{3.4.3}$$

The function $\tilde{\xi}^*$ will be used to construct the trial spaces for the velocities. Homogenizing boundary conditions if necessary it will be sufficient to construct these trial spaces as subspaces of $H_0^1(\Omega)$. Therefore according to Remark 3.4, we have defined the diminished index sets $\tilde{K}^*_{j,L,0}, \tilde{K}^*_{j,R,0}$ in (3.4.3) without the extreme indices $N-d, 2^j-N+1$ which correspond to those basis functions which facilitate reproduction of constants. Thus constructing $\tilde{\Xi}^*_j = \{\tilde{\xi}^*_{j,k} : k \in \tilde{K}^*_j\}$ as described in Section 3.3, and $\tilde{\Xi}^*_{j,0}$ as in Remark 3.4 relative to

$$\tilde{K}^*_{j,0} := \tilde{K}^*_{j,L,0} \cup \tilde{K}^*_{j,I} \ \cup \ \tilde{K}^*_{j,R,0},$$

we have

$$S(\tilde{\Xi}^*_{j,0}) \ \subset \ H_0^1([0,1]). \tag{3.4.4}$$

Moreover, note that

$$\# K^*_j = \# \tilde{K}^*_{j,0}. \tag{3.4.5}$$

More can be said which will be the first important observation (see also [29]). In fact, the following considerations prepare some necessary technical prerequisites for establishing the relation (3.2.17) for the modified wavelets adapted to the domain Ω. To this end, we define for any collection Ξ of functions $\partial\Xi := \{\frac{d}{dx}\xi : \xi \in \Xi\}$.

Proposition 3.6 *One has*

$$S(\Xi^*_j) = S(\partial \Xi_j), \tag{3.4.6}$$

as well as

$$S(\partial \tilde{\Xi}^*_{j,0}) \subseteq S(\tilde{\Xi}_j). \tag{3.4.7}$$

Moreover,

$$\left\{ g : g(x) = \int_0^x v(t)dt, \ x \in [0,1], \ v \in S(\tilde{\Xi}_j), \ \int_0^1 v(t)dt = 0 \right\} = S(\tilde{\Xi}^*_{j,0}). \tag{3.4.8}$$

Proof: In analogy to (3.3.12) let

$$\alpha^{*,L}_{r,m} := 2^{j/2} \left((2^j \cdot)^r, 2^{j/2} \tilde{\xi}^*(2^j \cdot -m) \right)_{L^2(I\!R)}, \quad m = -l, \ldots, N-1,$$

and analogously $\alpha^{*,R}_{r,m}, \tilde{\alpha}^{*,L}_{r,m}, \tilde{\alpha}^{*,R}_{r,m}$. Observe first that these coefficients satisfy discrete counterparts to the functional equations (3.2.3). In fact, one readily derives from (3.2.3) and (3.3.12) (see also (3.3.14)) that

$$\begin{aligned} \alpha^L_{r,m} - \alpha^L_{r,m-1} &= r\alpha^{*,L}_{r-1,m}, \quad m = -l+1, \ldots, N-1, \ r = 0, \ldots, d-1, \\ \alpha^R_{r,m-1} - \alpha^R_{r,m} &= r\alpha^{*,R}_{r-1,m}, \quad m = 2^j - N + 2, \ldots, 2^j + l, \ r = 0, \ldots, d-1, \end{aligned} \tag{3.4.9}$$

and

$$\begin{aligned} \tilde{\alpha}^{*,L}_{r,m+1} - \tilde{\alpha}^{*,L}_{r,m} &= r\tilde{\alpha}^L_{r-1,m}, \quad m = -l, \ldots, N-1, \ r = 0, \ldots, d, \\ \tilde{\alpha}^{*,R}_{r,m} - \tilde{\alpha}^{*,R}_{r,m+1} &= r\tilde{\alpha}^R_{r-1,m}, \quad m = 2^j - N + 1, \ldots, 2^j + l, \ r = 0, \ldots, d. \end{aligned} \tag{3.4.10}$$

As before we will write for the interior scaling functions $\xi^*_{j,k} := 2^{j/2}\xi^*(2^j \cdot -k), k \in K^*_{j,I}$, and analogously for $\tilde{\xi}_{j,k}, \tilde{\xi}^*_{j,k}$, $k \in K_{j,I}, \tilde{K}^*_{j,I}$, respectively. Utilizing (3.4.9) and (3.2.3), straightforward calculations yield

$$
\begin{aligned}
\frac{d}{dx}\xi^L_{j,k} &= \begin{cases} -2^j \; \xi^*_{j,N}, & k = N-d, \\ 2^j \; ((k-N+d)\; \xi^{*,L}_{j,k} - \alpha^L_{j,k-N+d,N-1}\xi^*_{j,N}), & k \in K_{j,L} \setminus \{N-d\}, \end{cases} \\
\frac{d}{dx}\xi_{j,k} &= 2^j(\xi^*_{j,k} - \xi^*_{j,k+1}), \quad k \in K_{j,I}, \qquad (3.4.11) \\
\frac{d}{dx}\xi^R_{j,k} &= \begin{cases} 2^j \; ((k-2^j+N-d)\; \xi^{*,R}_{j,k+1} + \alpha^R_{j,2^j-N+d-k,2^j-N+1}\xi^*_{j,2^j-N+1}), \\ \qquad k \in K_{j,R} \setminus \{2^j-N+d\}, \\ 2^j\xi^*_{j,2^j-N+1}, \quad k = 2^j - N + d. \end{cases}
\end{aligned}
$$

Similarly, combining (3.4.10) with (3.2.3), provides

$$
\begin{aligned}
\frac{d}{dx}\tilde{\xi}^{*,L}_{j,k} &= 2^j \left((k-N+d)\tilde{\xi}^L_{j,k-1} - \tilde{\alpha}^{*,L}_{k-N+d}\tilde{\xi}_{j,N}\right), \quad k \in \tilde{K}^*_{j,L,0}, \\
\frac{d}{dx}\tilde{\xi}^*_{j,k} &= 2^j(\tilde{\xi}_{j,k-1} - \tilde{\xi}_{j,k}), \quad k \in \tilde{K}^*_{j,I}, \qquad (3.4.12) \\
\frac{d}{dx}\tilde{\xi}^{*,R}_{j,k} &= 2^j \left(\tilde{\alpha}^{*,R}_{2^j-N+d+1-k,2^j-N}\tilde{\xi}_{j,2^j-N} - (2^j-N+d+1-k)\tilde{\xi}^R_{j,k}\right), \quad k \in \tilde{K}^*_{j,R,0}.
\end{aligned}
$$

One readily concludes now from (3.4.11) that

$$
S(\partial\Xi_j) \subseteq S(\Xi^*_j). \qquad (3.4.13)
$$

Moreover, to prove the converse inclusion note first that, again by (3.4.11), $\xi^{*,G}_{j,k} \in S(\partial\Xi_j)$, $k \in K^*_{j,G}, G \in \{L,R\}$. Thus, it suffices to confirm that $\xi^*_{j,k} \in S(\partial\Xi_j)$, $k \in K^*_{j,I}$. To this end, note that

$$
\begin{aligned}
\xi^*_{j,k} &= 2^{-j}\; \frac{d}{dx}\xi_{j,k} + \xi^*_{j,k+1} \;=\; \cdots \;=\; 2^{-j} \sum_{m=k}^{2^j-N} \frac{d}{dx}\xi_{j,m} + \xi^*_{j,2^j-N+1} \\
&= 2^{-j} \sum_{m=k}^{2^j-N} \frac{d}{dx}\xi_{j,m} + 2^{-j}\; \frac{d}{dx}\xi_{j,2^j-N+d},
\end{aligned}
$$

which proves (3.4.6).

Clearly, (3.4.7) follows from (3.4.12). Finally, to prove (3.4.8), let us denote for convenience also $\tilde{\xi}^*_{j,k} = \tilde{\xi}^{*,G}_{j,k}$, $k \in \tilde{K}^*_{j,G}, G \in \{L,R\}$, and analogously $\tilde{\xi}_{j,k}$. Then (3.4.12) can be briefly rewritten as

$$
\frac{d}{dx}\tilde{\xi}^*_{j,k} = \sum_{m \in K_j} c_{j,k,m}\; \tilde{\xi}_{j,m}. \qquad (3.4.14)
$$

Now let us denote the space defined on the left hand side of (3.4.8) by V_j. Since $S(\tilde{\Xi}^*_{j,0}) \subset H^1_0([0,1])$ one has $\int_0^1 \frac{d}{dx}\tilde{\xi}^*_{j,k}(x)\; dx = 0$, $k \in \tilde{K}^*_{j,0}$, so that, by (3.4.12), $S(\tilde{\Xi}^*_{j,0}) \subseteq V_j$. To see that $V_j \subseteq S(\tilde{\Xi}^*_{j,0})$, define $\mathbf{b} \in I\!\!R^{K_j}$ by

$$b_k := \int_0^1 \tilde{\xi}_{j,k}(x)\ dx,\ \ k \in K_j,$$

and set $\langle \mathbf{b} \rangle^\perp := \{\mathbf{c} \in I\!\!R^{K_j} :\ \mathbf{c}^T\mathbf{b} = 0\}$. In these terms one obviously has

$$V_j = \Big\{ g(x) = \sum_{k\in K_j} c_k \int_0^x \tilde{\xi}_{j,k}(t)\ dt :\ \mathbf{c} \in \langle \mathbf{b} \rangle^\perp \Big\}.$$

Since $\tilde{\xi}^*_{j,k} \in V_j, k \in \tilde{K}^*_{j,0}$, the vectors formed by the coefficients in (3.4.14) belong to $\langle \mathbf{b} \rangle^\perp$. Consider the $(\#\tilde{K}^*_{j,0}) \times (\#K_j)$ matrix $\mathbf{B}$ whose kth row contains the coefficients $c_{j,k,m}, m \in K_j$. It is easy to see that $\mathbf{B}$ has full rank $\#\tilde{K}^*_{j,0} = \#K_j - 1$. In fact, it is upper triangular and has nonvanishing entries on the diagonal. Hence $\langle \mathbf{b} \rangle^\perp = \{\mathbf{B}^T\tilde{\mathbf{c}} : \tilde{\mathbf{c}} \in I\!\!R^{\tilde{K}^*_{j,0}}\}$. Thus, whenever $v(x) := \sum_{k\in K_j} c_k \int_0^x \tilde{\xi}_{j,k}(t)\ dt \in V_j$, i.e., $\mathbf{c} \in \langle \mathbf{b} \rangle^\perp$, there exists $\tilde{\mathbf{c}} \in I\!\!R^{\tilde{K}^*_{j,0}}$ with $\mathbf{c} = \mathbf{B}^T\tilde{\mathbf{c}}$ which, in view of (3.4.14), just means $v = \sum_{k\in\tilde{K}^*_{j,0}} \tilde{c}_k\ \tilde{\xi}^*_{j,k}$. This completes the proof. □

Now let

$$\Upsilon_j = \{\eta_{j,k} : k \in J_j\}, \quad \tilde{\Upsilon}_j = \{\tilde{\eta}_{j,k} : k \in \tilde{J}_j = J_j\}$$

be the wavelet bases for the spaces $S(\Xi_j)$ and $S(\tilde{\Xi}_j)$, respectively. As mentioned above the wavelets with support in the interior of $[0,1]$ have the form (3.1.5). The remaining ones interfering with the end points of the interval have to be properly modified. Since at this point we do not have to make use of their explicit representation we refer to [17] for a listing of the corresponding coefficients. The important point here is to realize that instead of applying an analogous construction also for the spaces $S(\Xi^*_j)$ and $S(\tilde{\Xi}^*_{j,0})$, the bases Υ_j and $\tilde{\Upsilon}_j$ already determine the wavelets for these latter spaces. In fact, with the above preparations at hand we can show next that as in the shift-invariant case the wavelets corresponding to the spaces $S(\Xi^*_j)$ and $S(\tilde{\Xi}^*_{j,0})$ still arise from those for $S(\Xi_j)$ and $S(\tilde{\Xi}_j)$ by differentiation and integration, respectively.

Theorem 3.7 *Define*

$$\eta^*_{j,k}(x) := 2^{-j-2}\ \frac{d}{dx}\eta_{j,k}(x), \quad x \in [0,1],\ k \in J_j = J^*_j, \tag{3.4.15}$$

and

$$\tilde{\eta}^*_{j,k}(x) := -2^{j+2} \int_0^x \tilde{\eta}_{j,k}(t)dt, \quad ,x \in [0,1],\ k \in J_j. \tag{3.4.16}$$

Then the collections

$$\Upsilon^*_j := \{\eta^*_{j,k} : k \in J_j\}, \quad \tilde{\Upsilon}^*_j := \{\tilde{\eta}^*_{j,k} : k \in J_j\},$$

are biorthogonal wavelet bases, i.e.,

$$(\eta^*_{j,k}, \tilde{\eta}^*_{j,k'})_{L^2([0,1])} = \delta_{k,k'}, \quad k,k' \in J_j, \tag{3.4.17}$$

and

$$S(\Xi^*_j) = S(\Xi^*_{j-1}) \oplus S(\Upsilon^*_j), \quad S(\tilde{\Xi}^*_{j,0}) = S(\tilde{\Xi}^*_{j-1,0}) \oplus S(\tilde{\Upsilon}^*_j). \tag{3.4.18}$$

Moreover, one has

$$S(\partial\Upsilon_j) = S(\Upsilon^*_j), \quad S(\partial\tilde{\Upsilon}^*_j) = S(\tilde{\Upsilon}_j). \tag{3.4.19}$$

Proof: Clearly, (3.4.19) is an immediate consequence of (3.4.15) and (3.4.16). Moreover, since $S(\Xi_{j-1})$ contains all constant functions and since $S(\Xi_{j-1}) \perp S(\tilde{\Upsilon}_j)$, we have

$$\tilde{\eta}^*_{j,k}(1) = -2^{j+2}\int_0^1 \tilde{\eta}_{j,k}(y)dy = 0,$$

so that

$$\tilde{\eta}^*_{j,k}(0) = \tilde{\eta}^*_{j,k}(1) = 0, \quad k \in J_j. \tag{3.4.20}$$

Thus

$$S(\tilde{\Upsilon}^*_j) \subset H^1_0([0,1]), \tag{3.4.21}$$

and we can employ integration by parts to conclude from (3.4.15) and (3.4.16) that

$$(\eta^*_{j,k}, \tilde{\eta}^*_{j,k'})_{L^2([0,1])} = (\eta_{j,k}, \tilde{\eta}_{j,k'})_{L^2([0,1])}, \quad k,k' \in J_j,$$

so that (3.4.17) follows from the fact that Υ_j and $\tilde{\Upsilon}_j$ are, by construction, biorthogonal bases. Hence, in particular, the collections Υ^*_j and $\tilde{\Upsilon}^*_j$ consist of linearly independent functions, and

$$\dim(S(\Upsilon^*_j)) = \dim(S(\tilde{\Upsilon}^*_j)) = \#J_j. \tag{3.4.22}$$

On account of Remark 3.5 and (3.4.22), it remains to verify the relations

$$S(\tilde{\Upsilon}^*_j) \subset S(\tilde{\Xi}^*_{j,0}), \quad S(\Upsilon^*_j) \subset S(\Xi^*_j), \tag{3.4.23}$$

and

$$S(\tilde{\Upsilon}^*_j) \cap S(\tilde{\Xi}^*_{j-1,0}) = S(\Upsilon^*_j) \cap S(\Xi^*_{j-1}) = \{0\}. \tag{3.4.24}$$

To this end, (3.4.15) yields

$$\eta^*_{j,k} = 2^{-j-2}\frac{d}{dx}\eta_{j,k} \in S(\partial\Upsilon_j) \subset S(\partial\Xi_j),$$

so that the second part of (3.4.23) follows from (3.4.6). Furthermore, since $S(\Xi_j)$ contains constant functions so that, due to biorthogonality, the elements in $S(\tilde{\Upsilon}_j)$ have first order vanishing moments, i.e., $\int_0^1 \tilde{\eta}_{j,k}(x)dx = 0, k \in J_j$, the first relation in (3.4.23) follows from (3.4.16) and (3.4.8).

Now suppose that $g := \sum_{k\in J_j} c_k\tilde{\eta}^*_{j,k} \in S(\tilde{\Upsilon}^*_j) \cap S(\tilde{\Xi}^*_{j-1,0})$. By (3.4.17), one has $(g, \eta^*_{j,k})_{L^2([0,1])} = c_k$. On the other hand, since by (3.4.15) and the fact that $\tilde{\Xi}^*_{j-1,0} \subset H^1_0([0,1])$,

$$(\tilde{\xi}^*_{j-1,k}, \eta^*_{j,k'})_{L^2([0,1])} = -2^{-j-2}\Big(\frac{d}{dx}\tilde{\xi}^*_{j-1,k}, \eta_{j,k'}\Big)_{L^2([0,1])},$$

the relation $S(\tilde{\Xi}_{j-1}) \perp S(\Upsilon_j)$ ensures, on account of (3.4.7), that $c_k = 0, k \in J_j$, which confirms the second part of (3.4.18). Similarly, for $g := \sum_{k\in J_j} c_k \eta^*_{j,k} \in S(\Upsilon^*_j) \cap S(\Xi^*_{j-1})$ we obtain, in view of (3.4.17), $c_k = (g, \tilde{\eta}^*_{j,k})_{L^2([0,1])} = 0$ since, by (3.4.6), $S(\Xi^*_{j-1}) = S(\partial \Xi_{j-1})$ and by (3.4.16) $(\frac{d}{dx}\xi_{j-1,k'}, \tilde{\eta}^*_{j,k})_{L^2([0,1])} = -2^{j+2}(\xi_{j-1,k'}, \tilde{\eta}_{j,k})_{L^2([0,1])} = 0$. Here we have used that $\tilde{\Upsilon}^*_j \subset H^1_0([0,1])$ and that $S(\Xi_{j-1}) \perp S(\tilde{\Upsilon}_j)$ in the last step. This proves (3.4.18) and completes the proof. □

Thus, in addition to the bases $\Phi_j, \tilde{\Phi}_j$, defined according to (3.3.16), we can now define the modified bases as follows

$$\Phi^{(\nu)}_j := \Xi_j \otimes \cdots \otimes \Xi^*_j \otimes \cdots \otimes \Xi_j, \tag{3.4.25}$$

and likewise

$$\tilde{\Phi}^{(\nu)}_{j,0} := \tilde{\Xi}_{j,0} \otimes \cdots \otimes \tilde{\Xi}^*_{j,0} \otimes \cdots \otimes \tilde{\Xi}_{j,0}, \tag{3.4.26}$$

where Ξ^*_j and $\tilde{\Xi}^*_{j,0}$ are understood to occur at position ν for $\nu = 1, \ldots, n$. Corresponding wavelet bases are now obtained in a canonical fashion as

$$\Psi^{(\nu)}_j := \bigcup_{e\in E^*} \Psi^{(\nu)}_{e,j}, \quad \tilde{\Psi}^{(\nu)}_j := \bigcup_{e\in E^*} \tilde{\Psi}^{(\nu)}_{e,j}, \tag{3.4.27}$$

where for

$$\Upsilon_{e_\nu,j} := \begin{cases} \Xi_j, & \text{if } e_\nu = 0, \\ \Upsilon_j, & \text{if } e_\nu = 1, \end{cases}$$

and analogously $\Upsilon^*_{e_\nu,j}, \tilde{\Upsilon}_{e_\nu,j}, \tilde{\Upsilon}^*_{e_\nu,j}$,

$$\Psi^{(\nu)}_{e,j} := \Upsilon_{e_1,j} \otimes \cdots \otimes \Upsilon^*_{e_\nu,j} \otimes \cdots \otimes \Upsilon_{e_n,j}, \tag{3.4.28}$$

as well as

$$\tilde{\Psi}^{(\nu)}_{e,j} := \tilde{\Upsilon}_{e_1,j} \otimes \cdots \otimes \tilde{\Upsilon}^*_{e_\nu,j} \otimes \cdots \otimes \tilde{\Upsilon}_{e_n,j}. \tag{3.4.29}$$

Corollary 3.8 *In view of (3.4.19), one has*

$$S(\partial_\nu \tilde{\Psi}^{(\nu)}_j) = S(\tilde{\Psi}_j) \tag{3.4.30}$$

and by (3.4.4),

$$S(\tilde{\Psi}^{(\nu)}_j) \subset H^1_0([0,1]^n). \tag{3.4.31}$$

By our assumptions on the univariate generators $\xi, \tilde{\xi}$, it is clear that the above construction produces for every $\nu = 1, \ldots, n$ bases $\Phi^{(\nu)}$ which are biorthogonal to the $\tilde{\Phi}^{(\nu)}$. Although the regularity of the $\Phi^{(\nu)}_j$ is now lower than that of the Φ_j it still will be seen to suffice (see (3.3.1)) to give rise to stable multiscale bases in the sense of Section 2. In fact, we will show that these wavelet bases form Riesz bases for certain scales of Sobolev spaces. According to the results in Section 2, the main prerequisites can be formulated as follows.

Proposition 3.9 *Under the above assumptions there exists some $\gamma > 0$ such that for any $s < \gamma$*

$$\|v_j\|_{H^s(\Omega)} \lesssim 2^{js}\|v_j\|_{L^2(\Omega)}, \quad v_j \in S_j, \tag{3.4.32}$$

where S_j is any of the spaces $S(\Phi_j), S(\tilde{\Phi}_j), S(\Phi_j^{(\nu)}), S(\tilde{\Phi}_{j,0}^{(\nu)})$, $\nu = 1, \dots n$. In particular, $\gamma = t > 1$ (see (3.3.1)) for $S_j = S(\Phi_j), S(\tilde{\Phi}_j), S(\tilde{\Phi}_{j,0}^{(\nu)})$. Moreover, one has

$$\inf_{v_j \in S_j} \|v - v_j\|_{L^2(\Omega)} \lesssim 2^{-js}\ \|v\|_{H^s(\Omega)}, \quad v \in H^s,\ s \le d, \tag{3.4.33}$$

where $S_j = S(\Phi_j), S(\tilde{\Phi}_j)$ and $H^s = H^s(\Omega)$, or $S_j = S(\Phi_j)^\circ := S(\Phi_j) \cap L_0^2(\Omega)$ and $H^s = H^s(\Omega) \cap L_0^2(\Omega)$, or $S_j = S(\tilde{\Phi}_{j,0}^{(\nu)})$ and $H^s = H^s(\Omega) \cap H_0^1(\Omega)$, $\nu = 1, \dots, n$.

Proof: Since the claims are quite in keeping with what one expects under the given circumstances we will only indicate the main steps here and refer to [17] for a detailed proof of the above statements. As for the inverse estimate (3.4.32), it is shown in [15] with the aid of Fourier transforms that e.g.

$$\|2^{jn/2}\phi(2^j \cdot -k)\|_{H^s(\mathbb{R}^n)} \lesssim 2^{sj}.$$

Using $\|g\|_{H^s(\Omega)} = \inf_{f, f|\Omega = g} \|f\|_{H^s(\mathbb{R}^n)}$ this, in turn leads to

$$\|2^{jn/2}\phi(2^j \cdot -k)\|_{H^s(\Omega)} \lesssim 2^{sj}.$$

Then the same arguments as used in [15] can be employed to show that

$$\omega_d(v_j, t)_{L^2(\Omega)} \lesssim \left(\min\{1, t2^j\}\right)^s \|v_j\|_{L^2(\Omega)}, \quad v_j \in S_j,$$

where $\omega_d(\cdot, t)_{L^2(\Omega)}$ denotes the d-th order L^2-modulus of continuity (see (2.16)). As pointed out in Section 2, an inequality of this latter type is equivalent to (3.4.32) (see [14]).

The essence of the proof of the direct estimate is that polynomials of degree $d-1$ are contained in the spaces S_j under consideration and that biorthogonal bases consisting of compactly supported functions are available. One should keep in mind that, by Lemma 3.1 and Remark 3.2, exactness and regularity of the spaces $S(\tilde{\Phi}_j^{(\nu)})$ are at least as high as that of $S(\tilde{\Phi}_j)$. In fact, denoting by Q_j projectors of the form (2.14), and setting $\square_{j,k} := 2^{-j}(k + [0,1]^n), k \in \mathbb{Z}^n$, this fact can be used to show that for $v \in H^d(\Omega)$

$$\|v - Q_j v\|_{L^2(\square_{j,k})} \lesssim \inf_{P \in \Pi_{d-1}} \|v - P\|_{L^2(\Delta_{j,k})} \lesssim 2^{-jd}\|v\|_{H^d(\Delta_{j,k})},$$

where Π_{d-1} denotes the space of all polynomials of degree at most $d-1$ and $\Delta_{j,k}$ is a slightly larger domain than $\square_{j,k}$ whose diameter still remains proportional to 2^{-j}. Summing over the above estimates and using interpolation leads to (3.4.33). As for the direct estimate in (3.4.33) relative to $S_j = S(\tilde{\Phi}_{j,0}^*)$, one has to make also use of Remark 3.4. So it remains to comment on the case $S(\Phi_j)^\circ = S(\Phi_j) \cap L_0^2(\Omega)$. To this end, let

$$P_j q := \sum_{k \in I_j} (q, \tilde{\phi}_{j,k})_{L^2(\Omega)}\ \phi_{j,k}, \quad P_{-1} := 0, \tag{3.4.34}$$

and define

$$P_j^\circ q := P_j q - |\Omega|^{-1}\int_\Omega (P_j q)(x)dx, \qquad (3.4.35)$$

where, of course, in the present situation $|\Omega| = 1$. Observe that for $v \in L_0^2(\Omega)$

$$\begin{aligned} \inf_{v_j\in S(\Phi_j)^\circ}\|v-v_j\|_{L^2(\Omega)} &\le \|P_j^\circ v - v\|_{L^2(\Omega)} \lesssim \|P_j v - v\|_{L^2(\Omega)} + \left|\int_\Omega (P_j v)(x)\ dx\right| \\ &\le \|P_j v - v\|_{L^2(\Omega)} + \int_\Omega |(P_j v)(x) - v(x)|dx \qquad (3.4.36)\\ &\lesssim \|P_j v - v\|_{L^2(\Omega)} \le \inf_{v_j\in S(\Phi_j)}\left\{\|P_j v - v_j\|_{L^2(\Omega)} + \|v_j - v\|_{L^2(\Omega)}\right\} \\ &\lesssim \inf_{v_j\in S(\Phi_j)}\|v-v_j\|_{L^2(\Omega)}, \end{aligned}$$

since the P_j are uniformly bounded projectors. Now we can invoke the direct estimate (3.4.33) for the space $S_j = S(\Phi_j)$ to complete the proof. □

Now define the spaces

$$M_j := S(\Phi_j), \quad \vec{X}_j := S(\tilde{\Phi}_{j,0}^{(1)}) \times \cdots \times S(\tilde{\Phi}_{j,0}^{(n)}). \qquad (3.4.37)$$

We will state next direct and inverse estimates for the spaces $M_j, \vec{X}_j$ which in turn will lead to norm equivalences of the type (2.19), (2.22). Due to the biorthogonality and refinability of the bases $\Phi_j^{(\nu)}, \tilde{\Phi}_{j,0}^{(\nu)}$, for each $\nu \in \{0, \dots, n\}$ the mappings

$$Q_j^{(\nu)} v := \sum_{k\in I_j}\left(v, \phi_{j,k}^{(\nu)}\right)_{L^2(\Omega)} \tilde{\phi}_{j,k}^{(\nu)} \qquad (3.4.38)$$

are projectors and satisfy (2.10) where $\phi^{(0)} := \phi$ (see Remark 2.4). Thus, in particular, Proposition 2.5 applies which combined with the inverse and direct estimates in Proposition 3.9 yields the following estimates for the spaces M_j and $\vec{X}_j$.

Corollary 3.10 *Under the above assumptions one has for $-t \le s' \le s \le t$ the inverse inequalities*

$$\|\vec{v}_j\|_{H^s(\Omega)^n} \lesssim 2^{(s-s')j}\|\vec{v}_j\|_{H^{s'}(\Omega)^n}, \quad \vec{v}_j \in \vec{X}_j,$$

and

$$\|q_j\|_{H^s(\Omega)} \lesssim 2^{(s-s')j}\|q_j\|_{H^{s'}(\Omega)}, \quad q_j \in M_j.$$

Furthermore, the direct inequalities

$$\inf_{\vec{v}_j\in\vec{X}_j}\|\vec{v}-\vec{v}_j\|_{H^{s'}(\Omega)^n} \lesssim 2^{(s'-s)j}\|\vec{v}\|_{H^s(\Omega)^n}, \quad \vec{v} \in (H_0^1(\Omega)\cap H^s(\Omega))^n, \quad 0 \le s' \le s \le d,\ s' \le t,$$

$$\inf_{q_j\in M_j}\|q-q_j\|_{H^{s'}(\Omega)} \lesssim 2^{(s'-s)j}\|q\|_{H^s(\Omega)}, \quad q \in H^s(\Omega)^\circ,$$

hold where

$$H^s(\Omega)^\circ := \left\{q \in H^s(\Omega) : \int_\Omega q(x)dx = 0\right\}.$$

Finally, the projectors $Q_j^{(\nu)}, \nu = 1, \dots, n$, are uniformly bounded in H^1 (see [14, 15]).

Let

$$\Psi := \Phi_0 \bigcup_{j=1}^{\infty} \Psi_j$$

and define $\Psi^{(\nu)}, \tilde{\Psi}, \tilde{\Psi}^{(\nu)}$ in an analogous fashion. The main consequence of the above facts can be formulated now as follows.

Theorem 3.11 *Suppose that $\xi, \tilde{\xi}$ satisfy (3.3.1). Then the pairs $\Psi, \tilde{\Psi}$ and $\Psi^{(\nu)}, \tilde{\Psi}^{(\nu)}$, $\nu = 1, \dots n$, form biorthogonal Riesz bases for $L^2(\Omega)$. Moreover, the spaces $M_j, \vec{X}_j$ defined in (3.4.37) fulfill the LBB condition* (1.8) *uniformly in j.*

Proof: The Riesz basis properties are consequences of Theorem 2.3 and Proposition 3.9. The rest of the argument follows now exactly the reasoning for the case $\Omega = \mathbb{R}^n$. In fact, define the projectors $\vec{Q}_j$ from $\vec{X}$ onto $\vec{X}_j$ by

$$(\vec{Q}_j \vec{v})_\nu := Q_j^{(\nu)} v_\nu = \sum_{k \in I_j} \left(v_\nu, \phi_{j,k}^{(\nu)} \right)_{L^2(\Omega)} \tilde{\phi}_{j,k}^{(\nu)}, \quad \nu = 1, \dots, n,$$

so that for any $\vec{v} \in \vec{X}$

$$\vec{v}_\nu - (\vec{Q}_m \vec{v})_\nu = \sum_{j=m+1}^{\infty} \sum_{k \in \mathcal{J}_j} \left(v_\nu, \psi_{j,k}^{(\nu)} \right)_{L^2(\Omega)} \tilde{\psi}_{j,k}^{(\nu)} \quad =: \sum_{j=m+1}^{\infty} (\vec{w}_j)_\nu,$$

where $\mathcal{J}_j := J_j \times \cdots \times J_j$. Obviously, $\vec{w}_j \in S(\tilde{\Psi}_j^{(1)}) \times \cdots \times S(\tilde{\Psi}_j^{(n)})$. We readily obtain from (3.4.30) that $\operatorname{div} \vec{w}_j$ belongs to the complement $S(\tilde{\Psi}_j)$ of $S(\tilde{\Phi}_{j-1})$ in $S(\tilde{\Phi}_j)$. But by biorthogonality, we have

$$S(\tilde{\Psi}_j) \perp S(\Phi_{j-1})$$

which means here that

$$\left(\operatorname{div} \vec{w}_j, v_m \right)_{L^2(\Omega)} \;=\; b(\vec{w}_j, v_m) \;=\; 0, \quad v_m \in S(\Phi_m) = M_m, \; j > m. \tag{3.4.39}$$

The assertion follows now from Proposition 1.1 since, by Corollary 3.10, the projectors $\vec{Q}_j$ are bounded in $H^1(\Omega)^n$. □

In order to obtain a conforming discretization of the pressure, the corresponding trial space should be in $L_0^2(\Omega)$, i.e., instead of working with M_j one should use

$$M_j^\circ := \left\{ v_j \in S(\Phi_j) : \int_\Omega v_j(x)\, dx = 0 \right\}.$$

Since $M_j^\circ \subset M_j$ so that still $S(\tilde{\Psi}_j) \perp M_{j-1}^\circ$ it is clear that the spaces $M_j^\circ, \vec{X}_j$ also satisfy the LBB condition.

The perhaps simplest way to fulfill this constraint is to add an equation of the form

$$\sum_{k \in I_j} g_{j,k}\, q_k = 0$$

for the pressure coefficients $q_k, k \in I_j = K_j \times \cdots \times K_j$ where $g_{j,k} := \int_\Omega \phi_{j,k}(x)dx$. Due to the scale-invariant structure of the spaces the quantities $g_{j,k}$ can be easily retrieved from the values

$$g_{0,k} = \int_\Omega \phi_{0,k}(x)\ dx, \quad k \in I_0,$$

which can be efficiently computed with the aid of the techniques developed in [18] (see [23] for the available software and its documentation). Moreover, if the pressure coefficients are given in terms of the multiscale representation the additional equation involves only the quantities $g_{0,k}, k \in I_0$, since the integrals of the wavelets vanish. Hence, in particular, the projection (3.4.35) can be executed efficiently by transforming $q \in M_j$ first into the wavelet representation $q = \sum_{\ell=0}^{j} \sum_{k \in \mathcal{J}_\ell} d_{\ell,k}\psi_{\ell,k}$ and subtracting then $\sum_{k\in I_0} d_{0,k}g_{0,k}$. For a detailed description see [29].

Remark 3.12 *So far we have assumed that $\tilde{\xi}$ has at least the same degree of exactness as ξ and the above construction preserves this degree of exactness for both the spaces $S(\Xi_j)$ and $S(\tilde{\Xi}_j)$ adapted to Ω. We emphasize, however, that fixing ξ and thereby a possibly low degree of exactness for the pressure discretization, one could, in principle, choose $\tilde{\xi}$ to be exact of arbitrarily high degree $\tilde{d}-1 \geq d-1$, see [11] and Section 5 below for examples. This gives rise to a whole family of trial spaces for the velocities with respective higher degree of exactness. To ensure that the corresponding higher degree of accuracy shows its effect not only in the interior of Ω but also near the boundary, one has to modify the definition of the functions $\tilde{\xi}^L_{j,k}, \tilde{\xi}^R_{j,k}$ somewhat. We dispense here with the precise technicalities and refer to [17] for the details.*

3.5 Preconditioning

As pointed out in Section 2 (see also [14, 15] for details), the direct and inverse estimates in Proposition 3.9 and Corollary 3.10 entail a wider range of norm equivalences than just the L^2-Riesz basis property. This will play a crucial role for preconditioning the linear systems (1.12) arising from Galerkin discretizations based on the above trial spaces. Moreover, defining as above

$$Q_j\vec{v} := \left(Q_j^{(1)}v_1, \ldots, Q_j^{(n)}v_n\right)^T, \tag{3.5.1}$$

the results in Section 2 combined with Proposition 3.9 ensure that the mapping

$$\Lambda_s\vec{v} := \left(\Lambda_s^{(1)} \otimes \cdots \otimes \Lambda_s^{(n)}\right)\vec{v}, \tag{3.5.2}$$

where

$$\Lambda_s^{(\nu)} := \sum_{j=0}^{\infty} 2^{sj}(Q_j^{(\nu)} - Q_{j-1}^{(\nu)}),$$

satisfies

$$\|\Lambda_s\vec{v}\|_{H^{s'}(\Omega)^n} \sim \|\vec{v}\|_{H^{s+s'}(\Omega)^n}, \quad s+s' \in (-t+1, t). \tag{3.5.3}$$

Similarly, with P_j defined in (3.4.34),

$$\Theta_s := \sum_{j=0}^{\infty} 2^{js}\ (P_j - P_{j-1}) \tag{3.5.4}$$

satisfies

$$\|\Theta_s q\|_{H^{s'}(\Omega)} \sim \|q\|_{H^{s+s'}(\Omega)} \text{ for } s+s' \in (-t,t). \tag{3.5.5}$$

As a consequence, one has because of $t > 1$

$$\begin{aligned} \|P_j q\|_{H^{-1}(\Omega)} &\sim \|\Theta_{-1} P_j q\|_{L^2(\Omega)} = \|P_j \Theta_{-1} q\|_{L^2(\Omega)} \\ &\lesssim \|\Theta_{-1} q\|_{L^2(\Omega)} \sim \|q\|_{H^{-1}(\Omega)}. \end{aligned} \tag{3.5.6}$$

In fact, the P_j are uniformly bounded on $H^s(\Omega), s \in (-t,t)$ [14].

As for the relevance of the relations (3.5.3) and (3.5.5) for preconditioning, note that, since

$$\|\mathbf{A}_j \vec{v}_j\|_{H^{-1}(\Omega)^n} \sim \|\vec{v}_j\|_{H^1(\Omega)^n}, \quad \vec{v}_j \in \vec{X}_j,$$

one obtains for $\vec{w}_j = \Lambda_1 \vec{v}_j$, in view of (3.5.3),

$$\begin{aligned} \|\vec{w}_j\|_{L^2(\Omega)^n} &\sim \|\vec{v}_j\|_{H^1(\Omega)^n} \sim \|\mathbf{A}_j \vec{v}_j\|_{H^{-1}(\Omega)^n} \sim \|\Lambda_{-1}^* \mathbf{A}_j \vec{v}_j\|_{L^2(\Omega)^n} \\ &\sim \|\Lambda_{-1}^* \mathbf{A}_j \Lambda_{-1} \vec{w}_j\|_{L^2(\Omega)^n}. \end{aligned} \tag{3.5.7}$$

Expanding $\vec{w}_j$ in multiscale form and using the fact that the wavelets form a Riesz basis (3.5.7) is, on account of (2.6), easily seen to be equivalent to saying that

$$\text{cond}_2(\mathbf{D}_{-1} \mathbf{A}_{\Psi_j} \mathbf{D}_{-1}) = \mathcal{O}(1), \quad j \to \infty, \tag{3.5.8}$$

where $\mathbf{A}_{\Psi_j}$ is the stiffness matrix relative to the wavelet basis and $\mathbf{D}_{-1}$ is a diagonal matrix with diagonal entries $2^{-\ell}$ corresponding to the level ℓ.

Remark 3.13 *Since the matrices $\mathbf{A}_{\Psi_j}$ are usually not as sparse as the stiffness matrices $\mathbf{A}_{\Phi_j}$ relative to the fine scale basis functions one would rather compute and store the latter ones. Thus, given the sparse stiffness matrix relative to the fine scale bases $\tilde{\Phi}_j^{(\nu)}$, a change of bases realized by transformations of the form* (2.2) *followed by a symmetric diagonal scaling is a suitable preconditioner. By Remark 2.1, each application can be carried out in $\mathcal{O}(\dim \vec{X}_j)$ operations, see also the comments below at the end of Section 4.*

While the verification of the LBB condition made only use of the relations (3.4.15), (3.4.16) between the different wavelets, the above preconditioning strategy requires their explicit representation of the form (2.4) to form the corresponding multiscale transformations (2.5). Alternatively, the operators $\mathbf{A}_j$ could be preconditioned with the aid of a BPX scheme [5]. We have mentioned the former possibility here since an analogous scheme will be seen in the next section to work also for the Schur complement in the time dependent case.

The development so far offers a systematic way of constructing stable discretizations for the stationary Stokes problem for any spatial dimension. The remaining part of the paper is devoted to pointing out several additional advantageous features of this concept. The first one is concerned with preconditioning the systems arising from the time dependent case. The second one is the fact that the use of shift-invariant refinable functions offers very efficient ways of computing right hand sides and the entries of the stiffness matrices in a unified fashion which is again essentially independent of the spatial dimension [18].

4 The Time Dependent Case

The above construction produces pairs of trial spaces M_j, $\vec{X}_j$ satisfying the LBB condition uniformly in j. Thus the discrete problems (1.12) are uniquely solvable. Various strategies for iteratively solving saddle point problems of this type efficiently have recently been discussed in [4].

The upshot of the discussion there is that (1.12) can be solved iteratively with asymptotically optimal complexity (i.e., each iteration reduces the error by a factor which is bounded away from one independently of the meshsize), provided that the corresponding trial spaces satisfy the LBB condition uniformly and that good preconditioners for the blocks $\mathbf{A}_j$ and for the Schur complements $\mathbf{K}_j = \mathbf{B}_j \mathbf{A}_j^{-1} \mathbf{B}_j^*$ are available. As for $\mathbf{A}_j$ this is known to be the case. One may either employ a BPX strategy [5, 16, 26, 31] or a change of bases as suggested by (3.5.7) and (3.5.8) (see the remarks at the end of the previous section). Moreover, in the stationary case (1.3) (and ν not too small) the Schur complements $\mathbf{K}_j$ turn out to be already well conditioned [4].

However, this is no longer the case when dealing with fully discretized versions

$$\begin{pmatrix} \mathbf{A}_{\tau,j} & \mathbf{B}_j^* \\ \mathbf{B}_j & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{p}_j \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix} \tag{4.1}$$

of (1.1)(see [4]). Here the $\mathbf{B}_j$ are defined as in (1.11) while now $\mathbf{A}_{\tau,j}$ is defined by

$$\left(\mathbf{A}_{\tau,j} \vec{u}_j, \vec{v}_j\right) = a_\tau(\vec{u}_j, \vec{v}_j), \tag{4.2}$$

where

$$a_\tau(\vec{u}, \vec{v}) := (\vec{u}, \vec{v})_{L^2(\Omega)} + \tau a(\vec{u}, \vec{v}),$$

and $\tau = \nu\ \Delta t$ is related to the time step Δt and the Reynolds number. The condition number of the Schur complement increases with decreasing τ. So adhering to the discussion in [4], we will concentrate on the issue of preconditioning the corresponding Schur complements for the rest of this section.

The preconditioner proposed in [4] is based on approximately solving Neumann problems with respect to the Dirichlet form

$$D(v, w) := (\operatorname{grad} v, \operatorname{grad} w)_{L^2(\Omega)}$$

to produce a proper shift in the Sobolev scale. Its efficient realization seems to hinge, however, on the particular domain in an essential way. Here we propose a different approach which works whenever a suitable multiscale basis is available. It is again suggested by the remarks at the end of Section 3.5.

In the following let us denote by $\mathbf{K}_{\tau,j} := \mathbf{B}_j \mathbf{A}_{\tau,j}^{-1} \mathbf{B}_j^*$ the Schur complement for a fully implicit discretization in the time dependent case.

Theorem 4.1 *For* $\tau \geq 2^{-2j}$ *let* $\Theta_{s,j} := \Theta_s P_j = P_j \Theta_s$ *and*

$$\mathbf{C}_{\tau,j} := \tau I + \Theta_{-1,j} \Theta_{-1,j}^*. \tag{4.3}$$

Then for any $q \in M_j$ *one has*

$$\left(\mathbf{K}_{\tau,j} q, q\right)_{L^2(\Omega)} \sim \left(q, \mathbf{C}_{\tau,j}^{-1} q\right)_{L^2(\Omega)}, \tag{4.4}$$

i.e., $\mathbf{C}_{\tau,j}$ *gives rise to uniformly bounded condition numbers.*

Proof: Let $\mathbf{N}_j$ be defined by

$$D(u_j, q_j) = \langle \mathbf{N}_j^{-1} u_j, q_j \rangle.$$

It is well–known that $D(\cdot,\cdot)$ is $H^1(\Omega)^\circ$-elliptic so that

$$\langle \mathbf{N}_j^{-1} u_j, u_j \rangle \sim \|u_j\|^2_{H^1(\Omega)}, \quad u_j \in M_j. \tag{4.5}$$

On the other hand, by (3.5.5),

$$\begin{aligned} \|u_j\|^2_{H^1(\Omega)} &\sim \|\Theta_{1,j} u_j\|^2_{L^2(\Omega)} = (\Theta_{1,j} u_j, \Theta_{1,j} u_j)_{L^2(\Omega)} \\ &= (\Theta_{1,j}^* \Theta_{1,j} u_j, u_j)_{L^2(\Omega)}, \end{aligned}$$

which shows, in view of (4.5), that $\mathbf{N}_j^{-1}$ and $\Theta_{1,j}^* \Theta_{1,j}$ are spectrally equivalent. Thus $\mathbf{N}_j$ and $\Theta_{-1,j} \Theta_{-1,j}^*$ are spectrally equivalent. Since Ω is a convex polyhedral domain and since (3.5.6), Theorem 3.11 and Corollary 3.10 confirm that the assumptions made in [4] are satisfied here, the result in [4] ensures that $\tau I + \mathbf{N}_j$ is a preconditioner for which the above assertion holds. The claim follows now from the above spectral equivalence. □

The case $\tau \le 2^{-2j}$ can also be treated as in [4].

We conclude this section with a few comments on the application of $\mathbf{C}_{\tau,j}$. It suffices to discuss the term $\Theta_{-1,j} \Theta_{-1,j}^*$. By (2.19) and (3.5.5), one has

$$\begin{aligned} \left(\Theta_{-1,j} \Theta_{-1,j}^* q, q\right)_{L^2(\Omega)} &\sim \|q\|^2_{H^{-1}(\Omega)} \sim \sum_{l=0}^{j} 2^{-2l} \, \|(P_l^* - P_{l-1}^*) q\|^2_{L^2(\Omega)} \\ &= \sum_{l=0}^{j} 2^{-2l} \left((P_l - P_{l-1})(P_l^* - P_{l-1}^*) q, q\right)_{L^2(\Omega)} = \sum_{l=0}^{j} 2^{-2l} \left(\tilde{\mathbf{G}}^l \mathbf{q}^l, \mathbf{q}^l\right) \end{aligned}$$

where

$$\tilde{\mathbf{G}}^l = \left(\left(\tilde{\psi}_{l,k'}, \tilde{\psi}_{l,k}\right)_{L^2(\Omega)}\right)_{k',k \in J_l}, \quad (\mathbf{q}^l)_k = (q, \psi_{l,k})_{L^2(\Omega)},$$

and $(\cdot,\cdot)$ denotes the standard Euklidean inner product. By the stability of the bases $\{\tilde{\psi}_{l,k'}\}_{k' \in J_l}$ one has

$$\left(\tilde{\mathbf{G}}^l \mathbf{q}^l, \mathbf{q}^l\right) \sim (\mathbf{q}^l, \mathbf{q}^l),$$

so that the operator

$$\mathbf{R}_j q \; = \; \sum_{l=0}^{j} 2^{-2l} \sum_{k \in J_l} (q, \psi_{l,k})_{L^2(\Omega)} \psi_{l,k}$$

is spectrally equivalent to $\Theta_{-1,j} \Theta_{-1,j}^*$.

One can realize $\Theta_{-1,j}\Theta^*_{-1,j}$ in a slightly different way. Note that for any selfadjoint operator $\mathbf{L}$ on M_j and any $q \in M_j$

$$\Theta_{-1,j}\Theta^*_{-1,j}\mathbf{L}q \;=\; \Theta_{-1,j}\Theta^*_{-1,j}\mathbf{L}\Theta_{-1,j}(\Theta_{1,j}q).$$

Expanding q in terms of the multiscale basis $\{\psi_{l,k}\}_{l\le j,k\in J_l}$ and denoting the coefficients by $\tilde{\mathbf{q}} = \Big((q,\tilde\psi_{l,k})_{L^2(\Omega)}\ :\ k \in J_l,\ l = 0,\dots,j\Big)$, the term $\Theta^*_{-1,j}\mathbf{L}\Theta_{-1,j}$ simply involves a symmetric diagonal scaling applied to the matrix representation $\mathbf{L}_{\Psi_j}$ of $\mathbf{L}$ relative to the multiscale basis (see Remark 3.13). Of course, the stiffness matrix $\mathbf{L}_{\Psi_j}$ is not as sparse as the stiffness matrix $\mathbf{L}_{\Phi_j}$ relative to the fine scale basis. But it is not necessary to store $\mathbf{L}_{\Psi_j}$. In fact, since

$$\mathbf{L}_{\Psi_j} = \mathbf{T}_j^*\mathbf{L}_{\Phi_j}\mathbf{T}_j$$

where $\mathbf{T}_j$ denotes the corresponding multiscale transformation defined in (2.5), the application of $\mathbf{T}_j$ and $\mathbf{T}_j^*$ requires, in view of Remark 2.1, only $\mathcal{O}\Big(\dim S(\Phi_j)\Big)$ operations.

5 Construction of Trial Functions

It remains now to identify concrete examples of trial functions to which the recipe proposed in Section 3.4 applies. Thus, we have to find a dual pair $\xi, \tilde\xi$ satisfying (3.3.1) and (3.3.2) for some $t > 1$ and $d, \tilde d \ge 2$, say (where a smaller d would suffice for the stationary case). We focus here on the perhaps simplest approach based on the concrete biorthogonal wavelets constructed in [11]. We confine the discussion to the generators for the shift–invariant setting since the adaptation to the boundaries of Ω follows the lines in Section 3.3.

In the following, let N_m denote the shifted cardinal B–spline of order $m \in \mathbb{N}$ with knots at the integers. It may be defined as the mth order divided difference

$$N_m(x) := m[0,1,\dots,m](\cdot - x - \lfloor \tfrac{m}{2}\rfloor)_+^{m-1}$$

of the truncated power function or alternatively as the mth order convolution product of the characteristic function $N_1(x) := \chi_{[0,1)}(x)$ shifted by $\lfloor \frac{m}{2}\rfloor$. For various properties of B-splines and their practical evaluation one may consult any text book on spline functions such as [2]. In particular, N_m satisfies the refinement relation

$$N_m(x) = \sum_{k\in\mathbb{Z}} 2^{1-m}\binom{m}{k+\lfloor\frac{m}{2}\rfloor}\, N_m(2x-k). \tag{5.1}$$

For any $m \in \mathbb{N}$ a family of dual generators $\tilde N_{m,\tilde m}$ for $m + \tilde m$ even is constructed in [11] which realizes regularity and accuracy of arbitrary high order controlled by the parameter $\tilde m \in \mathbb{N}$. Let us denote in the following by ${}_m a_k$, ${}_{m,\tilde m}\tilde a_k$ the refinement coefficients of N_m and $\tilde N_{m,\tilde m}$, respectively. In view of (5.1), we have, in particular,

$${}_m\mathbf{a}(z) = 2^{1-m}\; z^{-\lfloor\frac{m}{2}\rfloor}(1+z)^m, \qquad {}_m a_k = 2^{1-m}\binom{m}{k+\lfloor\frac{m}{2}\rfloor}. \tag{5.2}$$

We recall from [11] that the corresponding biorthogonal wavelets are then given by

$$\begin{aligned} {}_{m,\tilde m}\psi(x) &= \sum_{k\in\mathbb{Z}} (-1)^k \; {}_{m,\tilde m}\tilde a_{1-k} \; N_m(2x-k), \\ {}_{m,\tilde m}\tilde\psi(x) &= \sum_{k\in\mathbb{Z}} (-1)^k \; {}_m a_{1-k} \; \tilde N_{m,\tilde m}(2x-k). \end{aligned} \tag{5.3}$$

It can be shown that

$$\mathrm{diam}\big(\mathrm{supp}({}_{m,\tilde m}\psi)\big) \;=\; \mathrm{diam}\big(\mathrm{supp}({}_{m,\tilde m}\tilde\psi)\big) \;=\; m+\tilde m-1.$$

The following concrete realizations of trial spaces are based on choosing the univariate refinable function ξ in Section 3.3 as the B–spline N_m. For the sake of simplicity we will point this out only for the case of two spatial variables $n=2$. As shown in Section 3, the higher dimensional case can be handled in exactly the same manner. In order to generate the velocity spaces by scaling functions with possibly small supports relative to their regularity we let, according to (3.2.6),

$$\tilde\phi := N_m \otimes N_m. \tag{5.4}$$

Thus, setting for any $\tilde m$ such that $m+\tilde m$ is even

$$\phi := \tilde N_{m,\tilde m} \otimes \tilde N_{m,\tilde m}, \tag{5.5}$$

$\phi, \tilde\phi$ form by our previous remarks a dual pair. Defining N_m^* as in Lemma 3.1, one readily concludes from (5.2) and (3.2.4) that

$$N_m^* = N_{m-1} \tag{5.6}$$

and

$$\tilde N^*_{m,\tilde m} = \tilde N_{m-1,\tilde m+1}. \tag{5.7}$$

This yields the following facts.

Proposition 5.1 *For ϕ and $\tilde\phi$ defined by* (5.4) *and* (5.5), *respectively, one has*

$$\begin{aligned} \tilde\phi^{(1)} &= N_{m+1} \otimes N_m \\ \tilde\phi^{(2)} &= N_m \otimes N_{m+1}. \end{aligned} \tag{5.8}$$

The corresponding wavelets are

$$\begin{aligned} \tilde\psi^{(1)}_{(0,1)} &= N_{m+1} \otimes \; {}_{m,\tilde m}\psi, \qquad \tilde\psi^{(1)}_{(1,1)} \;=\; {}_{m+1,\tilde m-1}\psi \otimes \; {}_{m,\tilde m}\psi, \\ \tilde\psi^{(1)}_{(1,0)} &= {}_{m+1,\tilde m-1}\psi \otimes N_m. \end{aligned}$$

A few comments on the choice of $m, \tilde m$ are in order. By (5.8), $m=2$ is the smallest possible order suggested by the above construction principle. The pressure spaces are then spanned by piecewise multilinear functions. Since in this case $m=d>1$ and $\phi \in H^t(\mathbb{R}^2)$ for some $t>1$, the previously discussed preconditioner for the Schur complement in

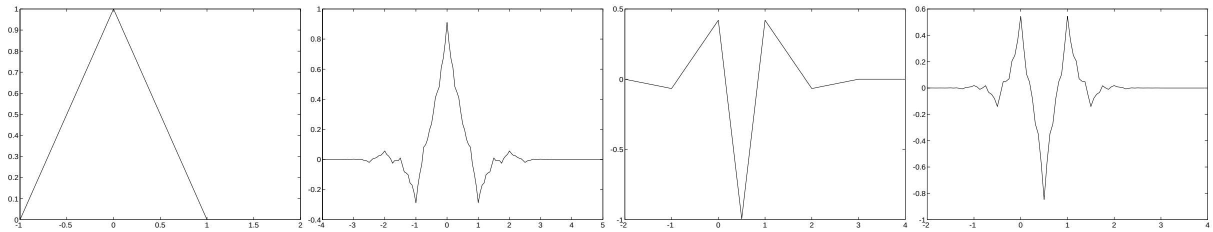


Figure 1: Generators and biorthogonal wavelets according to the dual pair $\phi = N_2$ and $\tilde{\phi} = \tilde{N}_{2,4}$.

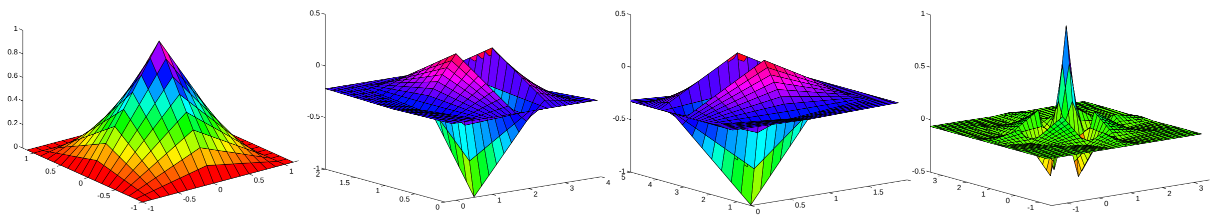


Figure 2: Bivariate biorthogonal wavelets generated by $N_2 \otimes N_2$.

the time dependent case applies. To ensure that the stability relations (2.19) hold for a sufficiently large range also for the velocity spaces, it suffices to choose $\tilde{m}$ so that $\tilde{N}_{2,\tilde{m}} \in H^{1+\epsilon}(I\!R)$ for some $\epsilon > 0$. It is known that $N_3, \tilde{N}_{3,3}$ form a dual pair of compactly supported refinable functions in $L^2(I\!R)$ (see [11], p. 548). This implies that $\tilde{N}_{3,3} \in H^{\varepsilon}(I\!R)$ for some $\varepsilon > 0$, see [9, 30]. By (5.7) and (3.2.3), $\tilde{N}_{2,4}$ is in $H^{1+\varepsilon}(I\!R)$, so that here $\tilde{m} = 4$ is sufficient.

Figure 1 exhibits the graph of N_2, $\tilde{N}_{2,4}$ and the corresponding biorthogonal wavelets ${}_{2,4}\psi$, ${}_{2,4}\tilde{\psi}$. The bivariate wavelets corresponding to $\tilde{\phi} = N_2 \otimes N_2$ are

$$\tilde{\psi}_{(0,1)} \;=\; N_2 \otimes {}_{2,4}\psi, \quad \tilde{\psi}_{(1,0)} \;=\; {}_{2,4}\psi \otimes N_2, \quad \tilde{\psi}_{(1,1)} \;=\; {}_{2,4}\psi \otimes {}_{2,4}\psi \tag{5.9}$$

displayed in Figure 2.

The graphs of the dual pairs N_2^*, $\tilde{N}_{2,4}^*$ and the corresponding biorthogonal wavelets ${}_{2,4}\psi_e^{(1)}$, ${}_{2,4}\tilde{\psi}_e^{(1)}$, $e \in E^*$, are shown in Figure 3.

Finally the functions in Proposition 5.1 for $m = 2$, $\tilde{m} = 4$ are displayed in Figure 4.

The functions spanning the velocity spaces and the corresponding wavelets in the case $m = 2$, $\tilde{m} = 4$ are given by

$$\vec{\phi}^1 = \begin{pmatrix} \tilde{\phi}^{(1)} \\ 0 \end{pmatrix}, \qquad \vec{\phi}^2 = \begin{pmatrix} 0 \\ \tilde{\phi}^{(2)} \end{pmatrix}, \tag{5.10}$$

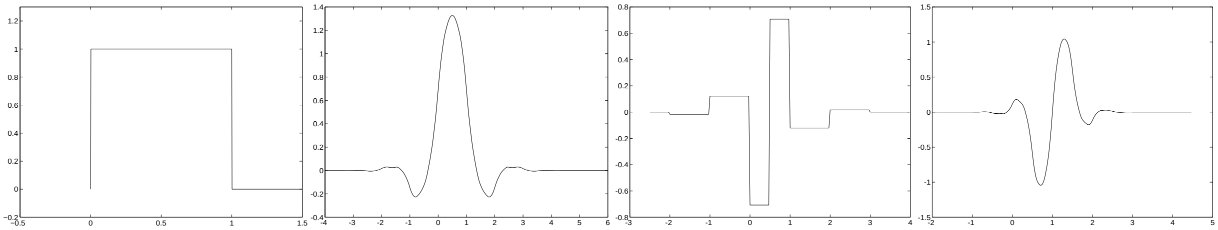


Figure 3: Generators and biorthogonal wavelets according to the dual pair $\phi = N_2^*$ and $\tilde{\phi} = \tilde{N}_{2,4}^*$.

and

$$\vec{\psi}^1 = \begin{pmatrix} \tilde{\psi}^{(1)}_{(0,1)} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{N}_{1,5} \otimes {}_{2,4}\tilde{\psi} \\ 0 \end{pmatrix}, \quad \vec{\psi}^2 = \begin{pmatrix} \tilde{\psi}^{(1)}_{(1,0)} \\ 0 \end{pmatrix} = \begin{pmatrix} {}_{1,5}\tilde{\psi} \otimes \tilde{N}_{2,4} \\ 0 \end{pmatrix},$$
$$\vec{\psi}^3 = \begin{pmatrix} \tilde{\psi}^{(1)}_{(1,1)} \\ 0 \end{pmatrix} = \begin{pmatrix} {}_{1,5}\tilde{\psi} \otimes {}_{2,4}\tilde{\psi} \\ 0 \end{pmatrix}, \quad \vec{\psi}^4 = \begin{pmatrix} 0 \\ \tilde{\psi}^{(2)}_{(0,1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{N}_{2,4} \otimes {}_{1,5}\tilde{\psi} \end{pmatrix}, \tag{5.11}$$
$$\vec{\psi}^5 = \begin{pmatrix} 0 \\ \tilde{\psi}^{(2)}_{(1,0)} \end{pmatrix} = \begin{pmatrix} 0 \\ {}_{2,4}\tilde{\psi} \otimes \tilde{N}_{1,5} \end{pmatrix}, \quad \vec{\psi}^6 = \begin{pmatrix} 0 \\ \tilde{\psi}^{(2)}_{(1,1)} \end{pmatrix} = \begin{pmatrix} 0 \\ {}_{2,4}\tilde{\psi} \otimes {}_{1,5}\tilde{\psi} \end{pmatrix},$$

where we have used superscripts to index vector-valued quantities.

6 Computation of Stiffness Matrices and Numerical Examples

The computation of stiffness matrices and right hand sides often take significantly more computational effort in classical finite element settings than the actual solution process. One principal advantage of employing ingredients of shift-invariant spaces is the fact that the usual quadrature techniques can be replaced by a completely different way of computing the relevant inner products. As shown in [18] the (up to round off exact) computation of integrals of products of arbitrarily many refinable functions or their derivatives can be reduced to solving an eigenvector-moment problem whose size depends only on the support of the involved refinable functions (3.1.1) but not on the discretization level, and which has to be solved only once. The fact that more than two factors are admitted allows one to treat right hand sides and non constant coefficients in a unified essentially dimension independent fashion for any desired degree of accuracy. Polynomial nonlinearities could be handled even exactly. We will exemplify this here for the above choice of refinable functions. An implementation of these methods is documented in [23]. To keep things simple we comment only on entries involving basis functions relative to interior indices since the basis functions adapted to the boundary are linear combinations of those latter ones where the coefficients are independent of the refinement level j.

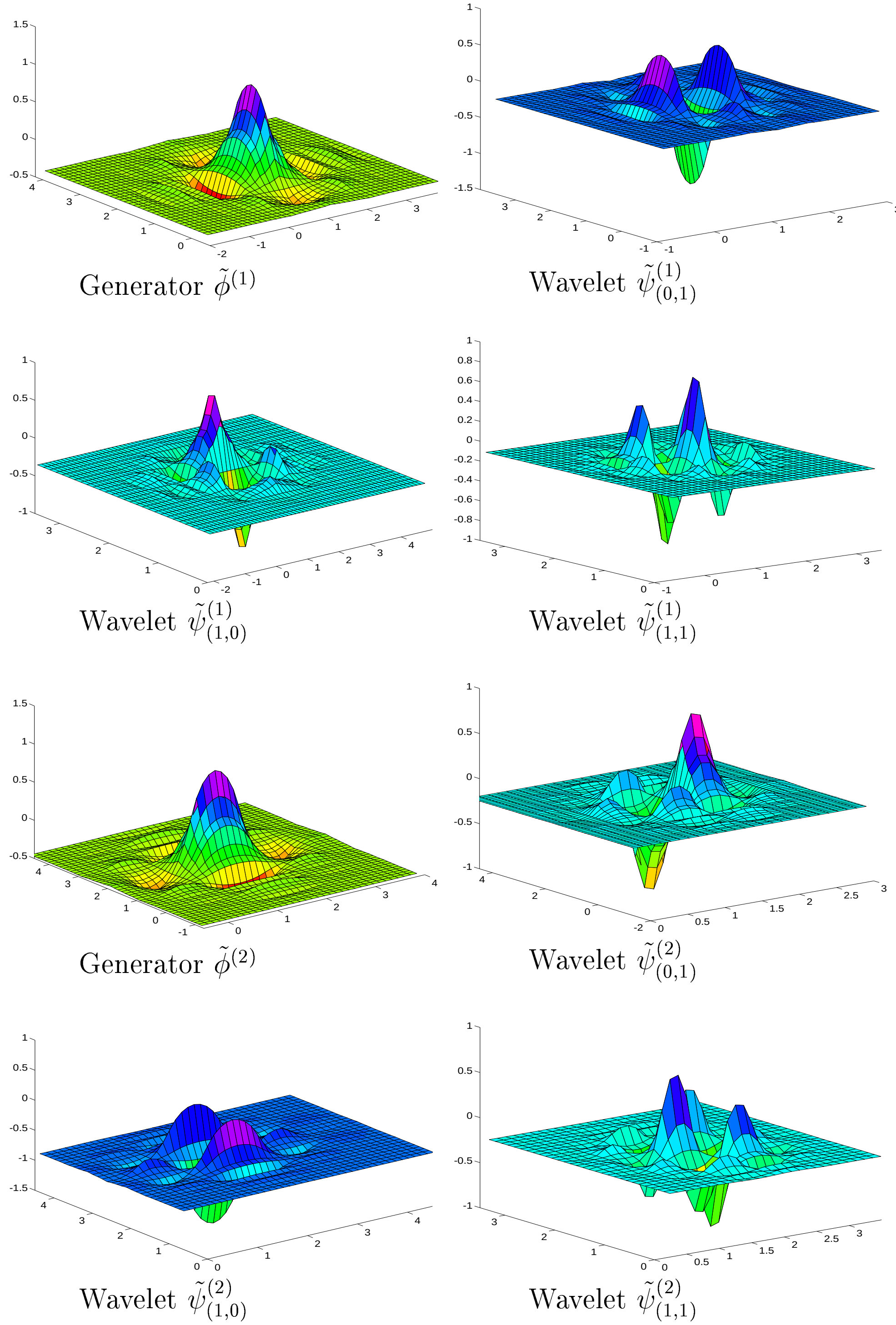


Figure 4: Modified functions according to Proposition 5.1.

As mentioned before only the stiffness matrices relative to the functions

$$\vec{\phi}^i_{j,k} \;=\; 2^j\vec{\phi}^i(2^j\cdot -k), \quad i=1,2, \quad k\in \mathbb{Z}^2,$$

need to be computed, where the $\vec{\phi}^i$, $i=1,2$, are defined in (5.10) and j denotes the highest refinement level. Specifically, one has to determine quantities of the form $a\Big(\vec{\phi}^{\nu}_{j,k}, \vec{\phi}^{\mu}_{j,k'}\Big)$, $b\Big(\vec{\phi}^{\nu}_{j,k}, \phi_{j,k'}\Big)$, i.e.

$$A^j_{\nu,\mu,k,k'} \;:=\; a\Big(\vec{\phi}^{\nu}_{j,k}, \vec{\phi}^{\mu}_{j,k'}\Big), \quad B^j_{\nu,k,k'} \;:=\; b\Big(\vec{\phi}^{\nu}_{j,k}, \phi_{j,k'}\Big). \tag{6.1}$$

First note that, by (5.10),

$$A^j_{\nu,\mu,k,k'} \;=\; 0 \quad \text{if } \nu\neq\mu. \tag{6.2}$$

The other entries are easily seen to be

$$A^j_{\nu,\nu,k,k'} \;=\; \left(\frac{\partial}{\partial x}\tilde{\phi}^{(\nu)}(\cdot-k), \frac{\partial}{\partial x}\tilde{\phi}^{(\nu)}(\cdot-k')\right)_{L^2(\mathbb{R}^2)} + \left(\frac{\partial}{\partial y}\tilde{\phi}^{(\nu)}(\cdot-k), \frac{\partial}{\partial y}\tilde{\phi}^{(\nu)}(\cdot-k')\right)_{L^2(\mathbb{R}^2)}, \tag{6.3}$$

which can efficiently be computed due to the above remarks. Using (3.2.11) and biorthogonality, provides

$$\begin{aligned} B^j_{\nu,k,k'} \;&=\; \Big(\text{div }\vec{\phi}^{\nu}_{j,k}, \phi_{j,k'}\Big)_{L^2(\mathbb{R}^2)} \;=\; \left(\frac{\partial}{\partial x_\nu}\tilde{\phi}^{(\nu)}_{j,k}, \phi_{j,k'}\right)_{L^2(\mathbb{R}^2)} \\ &=\; 2^{-j}\Big(\Delta_\nu\tilde{\phi}(\cdot-k), \phi(\cdot-k')\Big)_{L^2(\mathbb{R}^2)} \;=\; 2^{-j}\left(\delta_{k-e^\nu,k'}-\delta_{k,k'}\right). \end{aligned}$$

Hence the matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ have the following structure

$$\mathbf{A}_j = \begin{pmatrix} \mathbf{A}^j_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^j_2 \end{pmatrix} \qquad \mathbf{B}_j = \begin{pmatrix} \mathbf{B}^j_1 \\ \mathbf{B}^j_2 \end{pmatrix}, \tag{6.4}$$

where the blocks are given by

$$\mathbf{A}^j_i \;=\; \Big(A^j_{i,i,k,k'}\Big)_{k,k'} \text{ and } \mathbf{B}^j_i \;=\; \Big(B^j_{i,k,k'}\Big)_{k,k'}.$$

As for the right hand side of the linear system, one can approximate each component f_ν by a linear combination of some appropriately scaled refinable functions of the form $\sum_{k\in I_j} c_k\,\zeta_{j,k}$. A convenient choice is $\zeta = N_m$ for some $m\in\mathbb{N}$ so that highly accurate local quasi–interpolant schemes can be employed to determine the coefficients c_k (see [2]). It remains then to evaluate again inner products of the form $(\zeta_{j,k}, \tilde{\phi}^{(\nu)}_{j,k'})_{L^2(\Omega)}$ by the techniques in [18, 23]

Let us conclude this paper with some preliminary numerical experiments using those functions constructed in Section 3.4 and Section 5. As a first step we treat the *Driven Cavity Stokes Problem* in two and three space dimensions, which describes the flow of a viscous, incompressible fluid over a box. To be specific, we consider the system (6.5)

$$\begin{aligned} -\Delta\vec{u} + \text{grad } p \;&=\; \vec{f} \qquad \text{in } \Omega, \\ \text{div } \vec{u} \;&=\; 0 \qquad \text{in } \Omega, \\ \vec{u} \;&=\; \vec{g} \qquad \text{on } \Gamma, \end{aligned} \tag{6.5}$$

for $\Omega = [0,1]^n$, $n = 2, 3$, and the particular data exhibited in Figure 5.

$$\vec{f} = 0 \quad \text{in} \quad \Omega,$$

$$\vec{g} = \begin{cases} \binom{0}{0} & \text{on } \partial\Omega \setminus \{x_n = 1\}, \\ \binom{1}{0} & \text{on } \partial\Omega \cap \{x_n = 1\}. \end{cases}$$

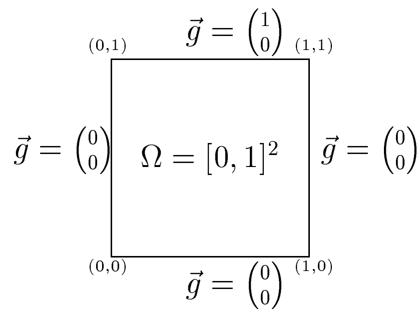


Figure 5: Particular data for the Driven–Cavity–Problem

To solve the weak problem (1.7) we employ stable pairs of trial spaces for several different generators displayed in Section 5. In this first implementation we treat the boundary conditions by homogenization. To retain the efficiency of computing inner products we represent the homogenizing function also as a linear combination of refinable functions (see [28]).

More extensive numerical tests for the time dependent problem will be reported in a forthcoming paper.

The saddle point problem (1.12) is solved here by the classical *Uzawa* algorithm using conjugate directions (see e.g. [3]). We view this also as a preliminary step to employ next the alternatives described in [4]. The linear systems arising in the Uzawa iteration are treated by means of a BPX–type preconditioned cg–method [5] whose realization for the present situation is described in [22, 28].

Our experiments cover the following choices of generators:

a) $\tilde{\phi} = N_2 \otimes N_2, \quad \phi = \tilde{N}_{2,4} \otimes \tilde{N}_{2,4}$ (see Section 5),

b) $\tilde{\phi} = N_3 \otimes N_3, \quad \phi = \tilde{N}_{3,3} \otimes \tilde{N}_{3,3}$.

U	Iteration steps of the Uzawa–algorithm
P	Total number of iteration steps for the pcg–iteration in the Uzawa–alg.
Unkn.	Total number of unknowns
Tol.	Tolerance
CPU	CPU time in seconds

Table 1: Abbreviations used in the subsequent tables.

$N_2 \otimes N_2$	**0.1**		**0.01**		**0.001**		**0.0001**		**0.00001**		**0.000001**		
Unkn./Tol.	**U**	**P**	**U**	**P**	**U**	**P**	**U**	**P**	**U**	**P**	**U**	**P**	**CPU**
21	2	3	3	4	3	4	3	4	3	4	3	4	0.007
133	2	40	5	77	6	89	9	119	12	144	13	151	0.504
645	1	38	3	71	7	132	11	182	17	234	20	253	3.406
2821	1	44	1	44	4	92	9	161	14	208	20	244	16.366
11781	1	49	1	49	2	63	5	102	10	153	16	189	63.894
48133	1	51	1	51	1	51	1	51	6	95	12	126	534.326
194565	1	51	1	51	1	51	1	51	1	51	7	76	646.017

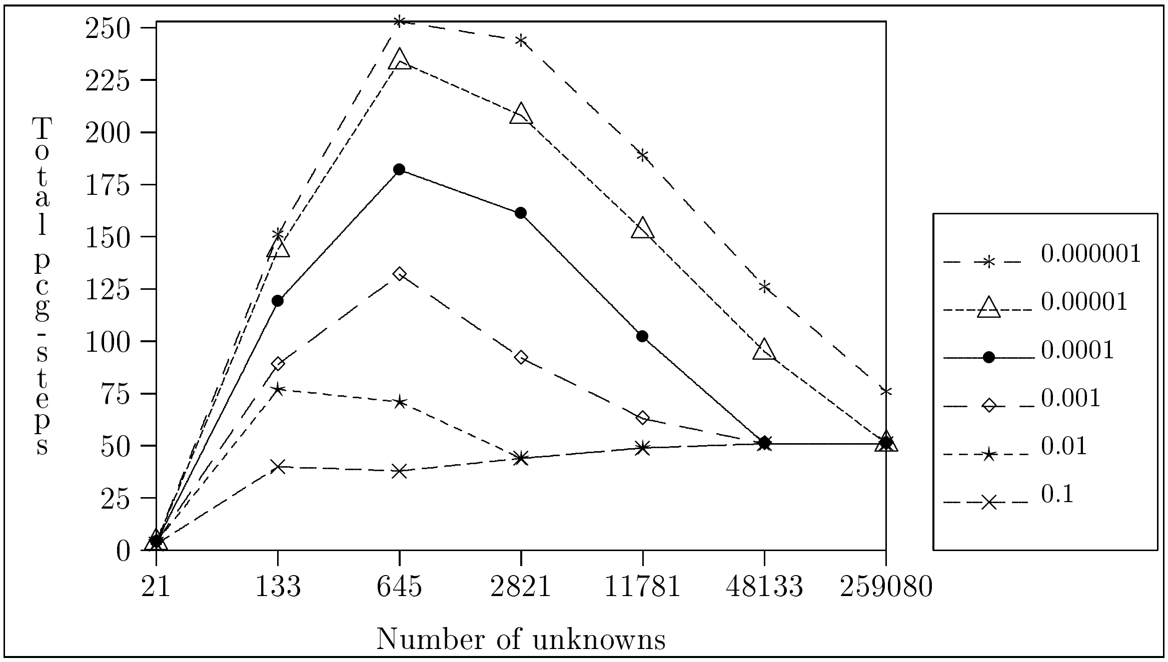
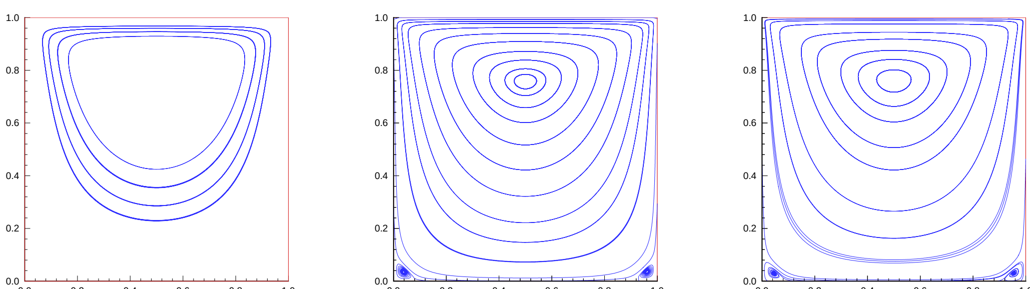
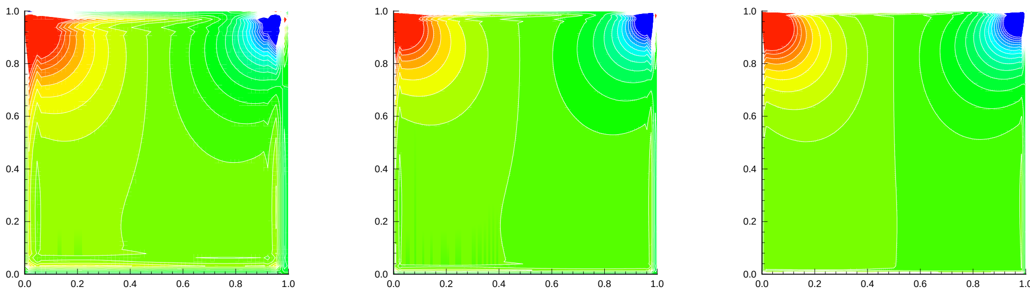


Figure 6: Number of iterations for $\tilde{\phi} = N_2 \otimes N_2$.



Level 5 Level 6 Level 7

Figure 7: Velocity for $\tilde{\phi} = N_2 \otimes N_2$.



Level 5 Level 6 Level 7

Figure 8: Pressure for $\phi = \tilde{N}_{2,4} \otimes \tilde{N}_{2,4}$.

$N_3 \otimes N_3$ **Unkn./Tol.**	**0.1** U	**0.1** P	**0.01** U	**0.01** P	**0.001** U	**0.001** P	**0.0001** U	**0.0001** P	**0.00001** U	**0.00001** P	**0.000001** U	**0.000001** P	**0.000001** CPU
8	2	3	2	3	2	3	2	3	2	3	2	3	0.004
96	3	64	7	129	11	184	15	232	17	252	19	270	0.252
560	1	52	7	208	14	369	22	522	29	623	40	748	5.393
2640	1	52	1	52	11	297	22	527	34	720	47	862	38.694
11408	1	48	1	48	1	48	14	301	27	502	40	636	180.762
47376	1	46	1	46	1	46	1	46	16	250	30	375	509.175
193040	1	44	1	44	1	44	1	44	1	44	18	162	1060.282

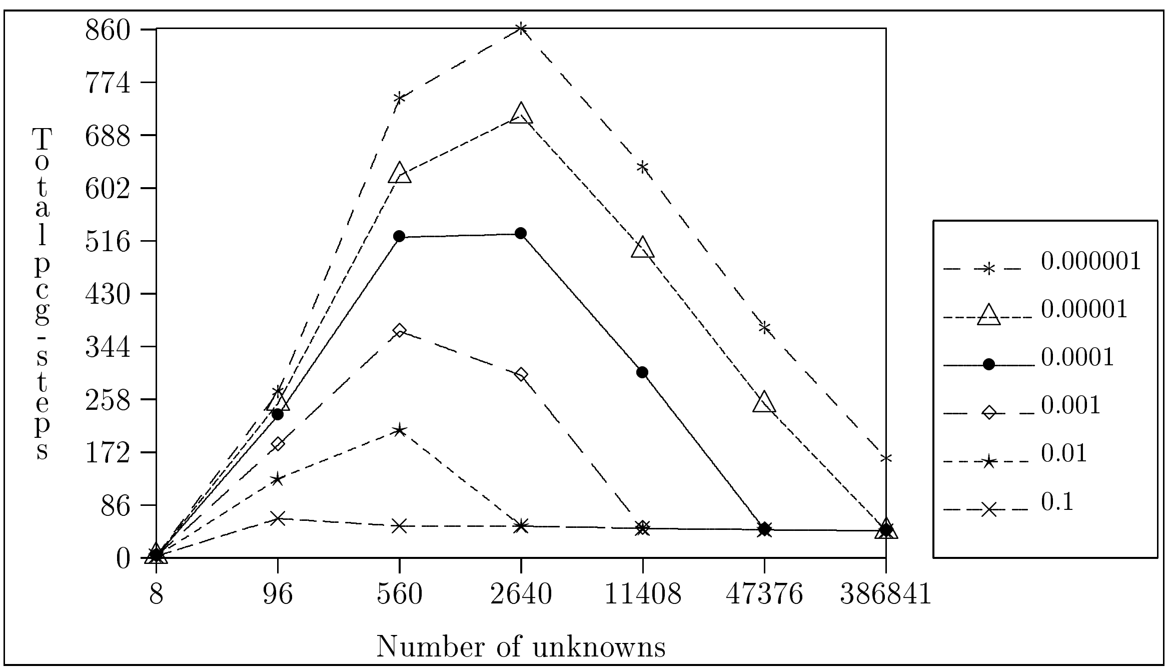
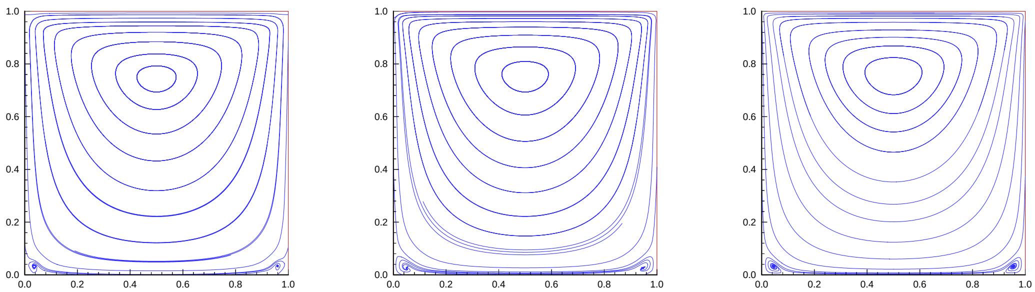
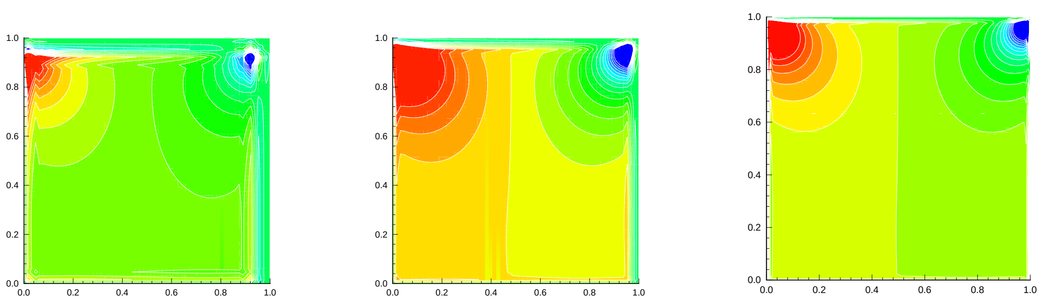


Figure 9: Number of iterations for $\tilde{\phi} = N_3 \otimes N_3$.



Level 5 Level 6 Level 7

Figure 10: Velocity for $\tilde{\phi} = N_3 \otimes N_3$.



Level 5 Level 6 Level 7

Figure 11: Pressure for $\phi = \tilde{N}_{3,3} \otimes \tilde{N}_{3,3}$.

A 3D Example

The construction of the trial spaces as well as the implementation of the algorithm is not restricted to any particular spatial dimension. In the following we document some results in $3D$.

$N_2 \otimes N_2 \otimes N_2$	**0.1**		**0.01**		**0.001**		**0.0001**		**0.00001**		**0.000001**		
Unkn./Tol.	**U**	**P**	**U**	**P**	**U**	**P**	**U**	**P**	**U**	**P**	**U**	**P**	**CPU**
81	5	12	7	16	8	18	9	19	10	20	10	20	0.024
1225	7	156	12	246	20	364	23	403	29	464	36	512	6.571
12825	1	47	9	231	19	433	31	626	39	727	46	794	213.457
116281	1	47	1	47	12	269	23	462	35	617	48	737	2299.035

Table 2: Number of iterations for $\tilde{\phi} = N_2 \otimes N_2 \otimes N_2$.

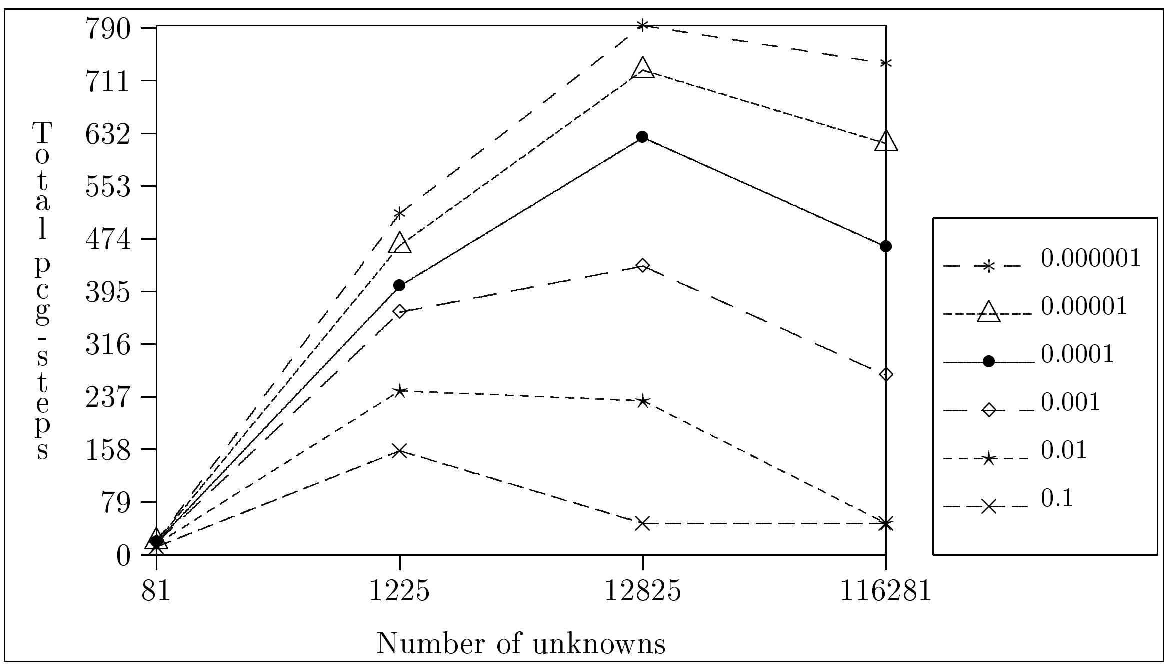


Figure 12: Number of iterations for $\tilde{\phi} = N_2 \otimes N_2 \otimes N_2$.

Figure 13 shows some streamlines for the 3D case. The different grey scales at the top corners of the cavity show the peaks in the pressure. As expected, one observes large positive values in the top left corner and negative values of large magnitude in the opposite corner. For a detailed description and further illustrations the reader is referred to [29].

It should be mentioned that by far most of the pcg–iterations recorded above occur during the first or first two Uzawa steps. The number of iterations drops then in all cases

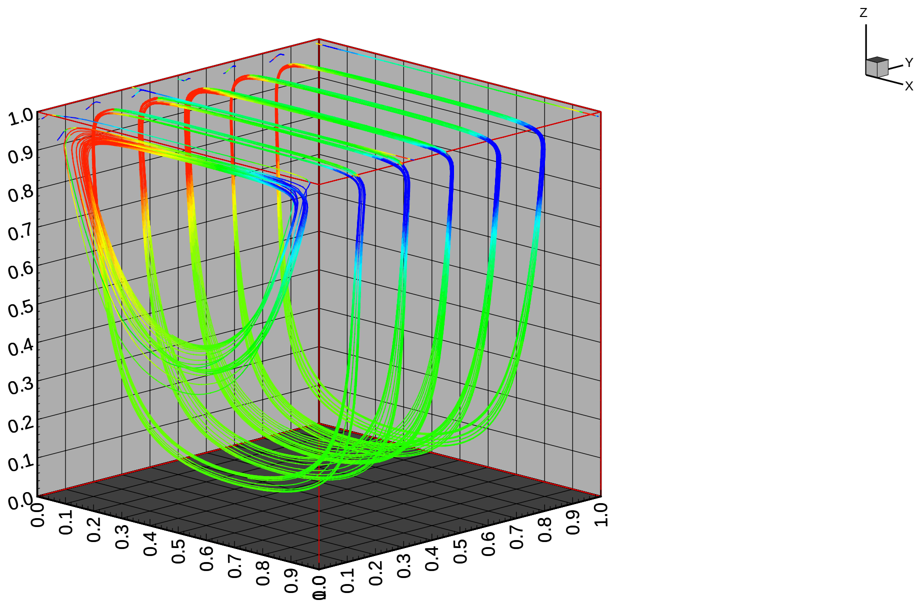


Figure 13: Streamlines in the 3D case. Colors show the amount of pressure.

below ten. This indicates that a better choice of starting values will help reducing the computational work significantly. It seems reasonable to use approximations from lower levels as starting values which we have not done yet. Nevertheless, even at the present stage of a rather crude implementation one observes that the overall number of iterations remains uniformly bounded. Each iteration involves an amount of computational work and storage which is proportional to the number of unknowns (see Remark 3.13). This is reflected also by the recorded CPU times where, however, larger jumps on higher levels relect the need for swapping data.

So far we have been interested mostly in the quality of the discretizations and the basic preconditioning effects. Preliminary comparisons with e.g. the Taylor-Hood finite element seem to show that, for instance, the secondary vortices in the lower corners are resolved at a somewhat earlier stage by the present approach.

Recent first experiences with a more systematic and sophisticated development of software tools tuned to the particular features e.g. of the multiscale transformations and data structures indicate a great potential for further speed up. A detailed account of these ongoing investigations will be given elsewhere.

References

[1] L. Andersson, N. Hall, B. Jawerth and G. Peters, *Wavelets on closed subsets of the real line*, in: *Topics in the Theory and Applications of Wavelets*, L.L. Schumaker and G. Webb (eds.), Academic Press, Boston, 1994, 1-61.

[2] C. de Boor, *A Practical Guide to Splines*, Springer, 1978.

[3] D. Braess, *Finite Elemente*, Springer, 1992.

[4] J.H. Bramble and J.E. Pasciak, *Iterative techniques for time dependent Stokes problems*, Preprint, 1994.

[5] J.H. Bramble, J.E. Pasciak and J. Xu, *Parallel multilevel preconditioners*, Math. Comp., **55** (1990), 1-22.

[6] A.S. Cavaretta, W. Dahmen and C.A. Micchelli, *Stationary Subdivision*, Memoirs of the American Mathematical Society **93**, # 453, 1991.

[7] C.K. Chui and E. Quak, *Wavelets on a bounded interval*, in: Numerical Methods of Approximation Theory, D. Braess and L.L. Schumaker (eds.), Birkhäuser, 1992, 1-24.

[8] A. Cohen, W. Dahmen and R.A. DeVore, *Multiscale decompositions on bounded domains*, preprint, 1995.

[9] A. Cohen and I. Daubechies, *A stability criterion for biorthogonal wavelet bases and their related subband coding scheme*, Duke Mathematical Journal, **68** (1992), 313-335.

[10] A. Cohen, I. Daubechies, B. Jawerth and P. Vial, *Multiresolution analysis, wavelets and fast algorithms on an interval*, C.R. Acad. Sci. Paris, **316** (1993), 417-421.

[11] A. Cohen, I. Daubechies and J.-C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure and Appl. Math., **45** (1992), 485-560.

[12] A. Cohen, I. Daubechies and P. Vial, *Wavelets on the interval and fast wavelet transforms*, Appl. Comp. Harm. Anal., **1** (1993), 54–81.

[13] W. Dahmen, *Some remarks on multiscale transformations, stability and biorthogonality*, in: Wavelets, Images and Surface Fitting, P.J. Laurent, A. Le Méhauté and L.L. Schumaker (eds.), A K Peters, Wellesley, 1994, 157–188.

[14] W. Dahmen, *Stability of multiscale transformations*, RWTH Aachen, Preprint 1994.

[15] W. Dahmen, *Multiscale Analysis, Approximation and Interpolation Spaces*, to appear in: Approximation Theory VIII, C.K. Chui and L.L. Schumaker (eds.), World Scientific Publishing, 1995.

[16] W. Dahmen and A. Kunoth, *Multilevel preconditioning*, Numer. Math., **63** (1992), 315-344.

[17] W. Dahmen, A. Kunoth and K. Urban, *Wavelets and boundary conditions for the Stokes Problem*, in preparation.

[18] W. Dahmen and C.A. Micchelli, *Using the refinement equation for evaluating integrals of wavelets*, Siam J. Numer. Anal., **30** (1993), 507–537.

[19] M. Fortin, *An analysis of the convergence of mixed finite element methods*, R.A.I.R.O. Anal. Numer., **11** R3 (1977), 341-354.

[20] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes–Equations*, Springer Berlin, 1986.

[21] A. Jouini and P.G. Lemarié–Rieusset, *Analyses multirésolutions biorthogonales et applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

[22] A. Kunoth, *Multilevel preconditioning — Appending boundary conditions by Lagrange multipliers*, Advances in Computational Mathematics **4**, No. 1,2 (1995), 145–170.

[23] A. Kunoth, *Computing refinable integrals — Documentation of the program — Version 1.1*, Technical Report ISC-95-02-Math, Texas A&M University, May 1995.

[24] P.G. Lemarié–Rieusset, *Analyses multi-résolutions non orthogonales, Commutation entre Projecteurs et Derivation et Ondelettes Vecteurs à divergence nulle*, Revista Mat. Iberoamericana, **8** (1992), 221–236.

[25] P.G. Lemarié, *Fonctions a support compact dans les analyses multi–résolutions*, Revista Mat. Iberoamericana, **7** (1991), 157-182.

[26] P. Oswald, *On discrete norm estimates related to multilevel preconditioners in the finite element method*, in: Constructive Theory of Functions, Proc. Int. Conf. Varna 1991, K.G. Ivanov, P. Petrushev, B. Sendov (eds.), Bulg. Acad. Sci., Sofia, 1992, 203-214.

[27] K. Urban, *On divergence–free wavelets*, Advances in Computational Mathematics **4**, No. 1,2 (1995), 51–82.

[28] K. Urban, *A Wavelet–Galerkin Algorithm for the Driven–Cavity–Stokes–Problem in Two Space Dimensions*, RWTH Aachen, Preprint 1994.

[29] K. Urban, *Multiskalenmethoden für das Stokes–Problem und angepaßte Wavelet–Basen*, Thesis (in German), RWTH Aachen, 1995.

[30] L.F. Villemoes, *Sobolev regularity of wavelets and stability of iterated filter banks*, in: Progress in wavelet analysis and applications, Proc. Int. Conf. Toulouse 1992, Y. Meyer, S. Roques (eds.) Frontieres, Paris 1993, 243–251.

[31] J. Xu, *Theory of multilevel methods*, Report AM 48, Department of Mathematics, Pennsylvania State University, 1989.

[32] H. Yserentant, *Old and new convergence proofs for multigrid methods*, Acta Numerica 1993, 285–326.

Wolfgang Dahmen
Institut für Geometrie und Praktische Mathematik
RWTH Aachen
Templergraben 55
52056 Aachen
Germany
e–mail: `dahmen@igpm.rwth-aachen.de`

Angela Kunoth
Weierstrass–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstr. 39
10117 Berlin
Germany
e–mail: `kunoth@wias-berlin.de`

Karsten Urban
Institut für Geometrie und Praktische Mathematik
RWTH Aachen
Templergraben 55
52056 Aachen
Germany
e–mail: `urban@igpm.rwth-aachen.de`