MIXED FINITE ELEMENT METHODS ON DISTORTED RECTANGULAR GRIDS [1]

Jian Shen

Communicated by R. E. Ewing, R. D. Lazarov and J. M. Thomas.

Institute for Scientific Computation
Texas A & M University
College Station, TX 77843-3404, U.S.A.

July 1994

AMS(MOS) subject classifications. 65N12, 65N30

Abstract

A new mixed finite element method on totally distorted rectangular meshes is introduced with optimal error estimates for both pressure and velocity. This new mixed discretization fits the geometric shapes of the discontinuity of the rough coefficients and domain boundaries well. This new mixed method also enables us to derive the optimal error estimates and existence and uniqueness of Thomas's mixed finite elements method on distorted rectangular grids [19]. The lowest order Raviart-Thomas mixed finite rectangular element method becomes a special case of both methods, when all the elements are degenerated to parallelograms.

1. Introduction

Let Ω be a bounded convex polygonal domain in R^2, with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. We consider the homogeneous Dirichlet-Neumann boundary elliptic problem:

$$\begin{array}{rcll} -\nabla\cdot(\mathcal{K}\nabla p) & = & f, & \text{in} \quad \Omega, \\ p & = & 0, & \text{on} \quad \Gamma_1, \\ \mathcal{K}\nabla p\cdot\mathbf{n} & = & 0, & \text{on} \quad \Gamma_2, \end{array} \tag{1.1}$$

where $\mathcal{K}$ is a 2×2 symmetric positive definite matrix, which is uniformly bounded below and above in Ω; $\mathbf{n}$ is the outward unit normal of $\partial\Omega$. Extension to inhomogeneous boundary condition problems is straight forward. If we introduce a dependent vector valued variable $\mathbf{u} = -\mathcal{K}\nabla p$, then, (1.1) is equivalent to the following first order partial differential equation system

$$\begin{array}{rcllll} \mathcal{K}^{-1}\mathbf{u} & = & -\nabla p, & \text{in} & \Omega, & \text{(a)} \\ \nabla\cdot\mathbf{u} & = & f, & \text{in} & \Omega, & \text{(b)} \\ p & = & 0, & \text{on} & \Gamma_1, & \text{(c)} \\ \mathbf{u}\cdot\mathbf{n} & = & 0, & \text{on} & \Gamma_2. & \text{(d)} \end{array} \tag{1.2}$$

In the simulation of fluid flow in porous media, such as groundwater contamination and petroleum reservoir simulation, (1.2) arises frequently in a system of partial differential equations. Here, p stands for the hydraulic pressure or pressure and $\mathbf{u}$ stands for the fluid velocity or Darcy velocity. (1.2a) represents Darcy's law and (1.2b) is the mass conservation law, which is one of the

[1] This research was supported in part by the Department of Energy under Contract No. DE–FG05–92ER25143, through the Institute for Scientific Computation, Texas A & M University.

fundamental law in porous media flow. f is the source term. In general, the conductivity term $\mathcal{K}$ is discontinuous for heterogeneous reservoirs and the shapes of the discontinuous lines can be arbitrary. For example, in the simulation of incompressible miscible displacement, the conductivity can be written by $\mathcal{K} = \frac{k}{\mu}$, where k is a tensor representing the permeability of the medium which is discontinuous due to the the heterogeneity of the media and μ represents the viscosity of the fluid. μ is a continuous function of both time and space variables, but may have a very sharp frontal change of values. According to the mass conservation law (1.2b), the velocity $\mathbf{u}$ must be continuous along the normal direction of a subdomain boundary, no matter whether $\mathcal{K}$ is continuous or not. Roughly speaking, the discontinuity of the permeability cancels the discontinuity of the normal directional derivatives of the pressure along the subdomain boundaries.

The mixed finite element discretizations provide reasonable approximations for such kind of problems but not the standard Galerkin type finite element ones, see Falk and Osborn [10]. In [7][8][17][18], on orthogonal non-uniform rectangular grids, superconvergence error estimates for both pressure and velocity are derived under certain local smoothness assumptions. Mixed finite element methods on orthogonal grids and triangular grids have been studied intensively by many colleagues. Surveys on the development can be found, e.g., [3][13][15]. According to our computational experience on orthogonal grids, no approximation property can be achieved if the discontinuous lines fall inside any elements. Therefore, simulation of heterogeneous reservoirs require distorted rectangular grids to give an accurate model of the reservoir geology with a moderate number of grid cells.

The mixed triangular finite element discretizations can approximate these discontinuous coefficient problems well [10]. But computational implementation on unstructured triangular grids is not ease, especially in three-dimensional case, not mention that so many unknowns associated with triangular cells. So even though mixed triangular finite element methods was introduced at the same time as orthogonal rectangular finite element methods [14], the former is not as popular as the latter.

There have been some considerations for mixed finite element methods on distorted rectangular grids. Thomas [19] defined non-confirming velocity spaces without showing stability and approximation properties. Then, either element-wise mass balance may not be guaranteed on any distorted rectangular elements or the divergence of the discrete velocity spaces are not in the corresponding discrete pressure spaces. Farmer et. al. [11] and Russell [16] using Thomas's definition have obtained very encouraging computational results. In [16], using a finite volume approach by means of Thomas's finite element velocity space definition on distorted grids, Russell obtained superconvergence for both pressure and velocity in computational experiments. In [11][16], the velocity space is continuous in the normal direction on the element interfaces, thus, mass balance property is retained element-wise, but the divergence of the velocity space is no longer in the pressure space. Then, Russell [16] conjectured that the necessary and sufficient condition for mixed methods—*B-B inf-sup* condition to be satisfied only in the limit sense: when h tends to zero, all the irregular elements are almost parallelograms.

In this paper, we develop a mixed finite element method which is a natural and compact generalization of the lowest order Raviart-Thomas mixed finite orthogonal rectangular element method with stability and optimal error estimates. This method can be extended to three-dimensional case. Due to the complexity of the shapes of distorted cubes, we shall discuss three-dimensional case in another paper [5]. By our new mixed finite element discretization, we then enable to prove the Thomas's method on distorted grids [19] and Russell's conjecture about the *B-B inf-sup* condition

[16], if h is sufficiently small. Comparing with the lowest order mixed triangular discretization, we reduce the number of unknowns significantly on relatively well structured grids with the some order of accuracy. We feel that these two are practical methods for both numerical mathematicians and engineers from either theoretical or application point of views. To the author's knowledge, this is the first time to introduce such a method with solid analysis. According to [20], Professor Thomas has a way of proof for this numerical method in a totally different approach but he has never published his result before.

The remainder of this paper is organized as follows: We define the weak form of (1.2) in §2 with some notations for Sobolev spaces, the partition of the domain and some assumptions. In §3, we list some of the properties of bilinear and the contravariant Piola transformations. The mixed finite element spaces are defined in §4. In §5, projections and approximation properties of projections are discussed. The error estimates are presented in §6. The theoretical support for Thomas's definition [19] and Russell's conjecture about the *B-B inf-sup* condition [16] is §7. Finally, in §8, we give some conclusion remarks for implementations.

2. The Weak Formulations and Assumptions

Denote by $(\cdot,\cdot)$ the usual $L^2(\Omega)$ or $(L^2(\Omega))^2$ inner product. The spaces $H^k(\Omega)$, for k a positive integer, will be the usual Hilbert spaces equipped with the norms

$$\|\phi\|_k^2 = \sum_{|\alpha|\le k} \|\partial^\alpha \phi\|_0^2 = \sum_{|\alpha|\le k} (\partial^\alpha \phi, \partial^\alpha \phi). \tag{2.1}$$

The $H(\mathrm{div};\Omega)$ is defined by

$$H(\mathrm{div};\Omega) \equiv \left\{ \mathbf{v} \in (L^2(\Omega))^2 : \nabla\cdot\mathbf{v} \in L^2(\Omega) \right\},$$

equipped with the graph norm

$$\|\mathbf{v}\|_{H(\mathrm{div})}^2 = \|\mathbf{v}\|_0^2 + \|\nabla\cdot\mathbf{v}\|_0^2. \tag{2.2}$$

The Sobolev spaces $W_{k,\infty}(\Omega)$, for k a positive integer, are equipped with the norms

$$\begin{aligned} \|\phi\|_{k,\infty} &= \max_{|\alpha|\le k} \|\partial^\alpha \phi\|_\infty, \\ \|\phi\|_\infty &= \text{ees} \sup_\Omega |\phi|. \end{aligned}$$

Denote

$$\begin{aligned} &\mathbf{V} = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v}\cdot\mathbf{n} = 0, \text{ on } \Gamma_2\}, \\ &W = \begin{cases} L^2(\Omega), & \text{if } \Gamma_1 \ne \emptyset, \\ L^2(\Omega)\backslash R, & \text{if } \Gamma_1 = \emptyset. \end{cases} \end{aligned} \tag{2.3}$$

For solution of (1.2) and $\mathbf{v} \in \mathbf{V}$, applying the Green's formula and the homogeneous boundary condition, we obtain that

$$-(\nabla p, \mathbf{v}) = (p, \nabla\cdot\mathbf{v}) - \int_{\partial\Omega} p\mathbf{v}\cdot\mathbf{n} ds = (p, \nabla\cdot\mathbf{v}).$$

The mixed weak form of (1.2) is to seek a pair $(\mathbf{u}, p) \in \mathbf{V}\times W$ such that

$$\begin{aligned} (\mathcal{K}^{-1}\mathbf{u}, \mathbf{v}) &= (p, \nabla\cdot\mathbf{v}), & \mathbf{v} \in \mathbf{V}, \\ (\nabla\cdot\mathbf{u}, w) &= (f, w), & w \in W. \end{aligned} \tag{2.4}$$

To define a finite element method, we need a partition $\mathcal{T}_h$ of Ω into element K. For $K \in \mathcal{T}_h$, K is distorted rectangular but all its four edges are straight lines (see Figure 1.). If any edges of K, say, e, is on the boundary $\partial\Omega$, either Dirichlet or Neumann condition is imposed but not the both. Denote by $|K|$ the area of K. We require the quasi-regularity of the mesh, i.e., there are $1 > h > 0$ and $C^* \geq C_* > 0$ independent of h such that

$$C_* h^2 \leq |K| \leq C^* h^2, \qquad \forall\, K \in \mathcal{T}_h. \tag{2.5}$$

Note that we require that $K \in \mathcal{T}_h$ be convex to insure that the Jacobians (defined in §3) are no-singular. Through out the rest of this paper, C will be used to stand for a positive constant independent of h.

We assume that the boundaries of different rock types or regions of different permeability, i.e., the discontinuous lines are combination of finite many straight lines so that we can make grid lines coincide with the discontinuous lines. Denote by Γ the set of points where the coefficient $\mathcal{K}$ is discontinuous, adopted from [7][8][17][18] we make the following assumptions for solutions of (2.4).

<u>Assumption 2.1.</u>

1. *Γ consists of only line segments such that a distorted mesh can be formed on Ω to locate Γ on grid lines.*

2. *All the entries of $\mathcal{K}$ are smooth in $\Omega\backslash\Gamma$, say, in $W_{1,\infty}(\Omega\backslash\Gamma)$.*

3. *the pressure solution $p \in H^1(\Omega) \bigcap H^2((\Omega\backslash\Gamma)$. The one side normal and tangential derivatives of p on Γ are well defined and belong to $L^2(\Gamma)$, such that the velocity, $\mathbf{u}$ is continuous in the normal direction of Γ. In addition, we require that $\mathbf{u} \in (H^1(\Omega))^2$ and $\nabla \cdot \mathbf{u} \in H^1(\Omega)$.*

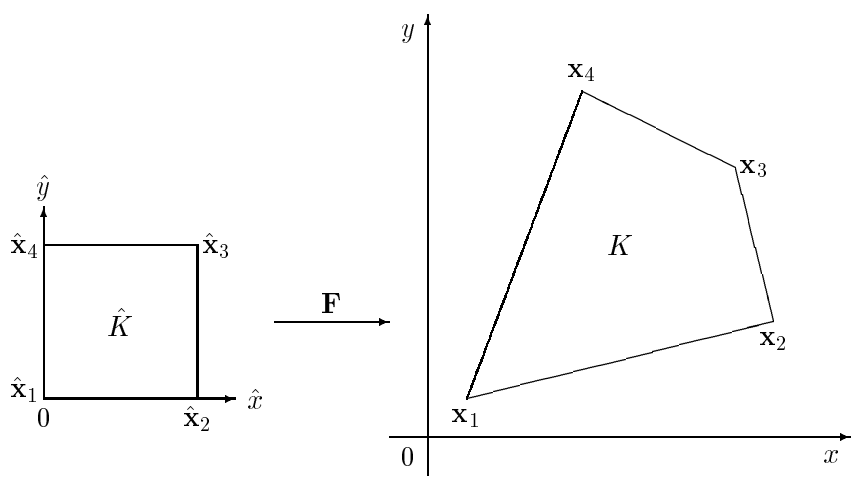

Figure 1. *The bilinear mapping.*

3. Properties Related to a Change of Variables and the Bilinear Transformation

Denote

$$\mathbf{x} = (x, y)^t, \qquad \hat{\mathbf{x}} = (\hat{x}, \hat{y})^t.$$

Define

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^t, \qquad \hat{\nabla} = (\frac{\partial}{\partial \hat{x}}, \frac{\partial}{\partial \hat{y}})^t.$$

We may also use $f_x = \partial_x f = \dfrac{\partial f}{\partial x}$ to stand for the partial derivative of f to the variable x.

Let $\mathbf{x}_i = (x_i, y_i)^t$, $i = 1, 2, 3, 4$ be the four vertices of element K counted anti-clockwise, and $\hat{\mathbf{x}}_i$, $i = 1, 2, 3, 4$ be the four vertices of the unit square $\hat{K}$ (see Figure 1.),

$$\hat{\mathbf{x}}_1 = (0, 0)^t, \quad \hat{\mathbf{x}}_2 = (1, 0)^t, \quad \hat{\mathbf{x}}_3 = (1, 1)^t, \quad \hat{\mathbf{x}}_4 = (0, 1)^t. \tag{3.1}$$

Let $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, the bilinear transformation, $\mathbf{F} : \hat{K} \to K$, is

$$\mathbf{x} = \mathbf{F}(\hat{\mathbf{x}}) = \mathbf{x}_1 + \mathbf{x}_{21}\hat{x} + \mathbf{x}_{41}\hat{y} + (\mathbf{x}_{32} - \mathbf{x}_{41})\hat{x}\hat{y}, \tag{3.2}$$

satisfying

$$\mathbf{x}_i = \mathbf{F}(\hat{\mathbf{x}}_i), \qquad i = 1, 2, 3, 4. \tag{3.3}$$

Denote by $\mathcal{J}$ the correspondent Jacobi matrix, and $J = |\det \mathcal{J}|$, the absolute value of the Jacobian,

$$\mathcal{J} = \begin{pmatrix} x_{\hat{x}} & x_{\hat{y}} \\ y_{\hat{x}} & y_{\hat{y}} \end{pmatrix}, \quad J = |\det \mathcal{J}| = x_{\hat{x}} y_{\hat{y}} - x_{\hat{y}} y_{\hat{x}} > 0, \tag{3.4}$$

and

$$\mathcal{J}^{-1} = \begin{pmatrix} \hat{x}_x & \hat{x}_y \\ \hat{y}_x & \hat{y}_y \end{pmatrix} = \frac{1}{J} \begin{pmatrix} y_{\hat{y}} & -x_{\hat{y}} \\ -y_{\hat{x}} & x_{\hat{x}} \end{pmatrix}. \tag{3.5}$$

Note that, under the bilinear transformation (3.2), J is a linear function of $\hat{x}$ and $\hat{y}$,

$$\begin{aligned} J &= (x_{21}y_{41} - x_{41}y_{21}) + (x_{21}y_{32} - x_{32}y_{21} + x_{41}y_{21} - x_{21}y_{41})\hat{x} \\ &\quad + (x_{32}y_{41} - x_{41}y_{32})\hat{y} \\ &= 2[\Delta_{124} + (\Delta_{123} - \Delta_{124})\hat{x} + (\Delta_{234} - \Delta_{123})\hat{y}] \\ &= \alpha + \beta\hat{x} + \gamma\hat{y}. \end{aligned} \tag{3.6}$$

In (3.6), Δ_{ijk} is the area of the triangle of $\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k$. The area of element K is

$$|K| = \int_K dK = \int_{\hat{K}} J d\hat{K} = \int_0^1 \int_0^1 J d\hat{x} d\hat{y} = \alpha + \frac{\beta}{2} + \frac{\gamma}{2}. \tag{3.7}$$

The Piola transformation $\mathcal{P}_K$, [3] [19], associated with the change of the variables (3.2) is defined by

$$\mathbf{u} = \mathcal{P}_K \hat{\mathbf{u}} = \frac{1}{J} \mathcal{J} \hat{\mathbf{u}}. \tag{3.8}$$

By the chain rule,

$$\nabla p = \mathcal{J}^{-t} \hat{\nabla} \hat{p}, \qquad \hat{p} = p(\mathbf{F}(\hat{\mathbf{x}})). \tag{3.9}$$

Since

$$\int_K f dK = \int_{\hat{K}} \hat{f} J d\hat{K}, \qquad \hat{f} = f(\mathbf{F}(\hat{\mathbf{x}})), \tag{3.10}$$

letting $\mathbf{v} = \frac{1}{J}\mathcal{J}\hat{\mathbf{v}}$, we have

$$\int_K \nabla p \cdot \mathbf{v} dK = \int_{\hat{K}} \hat{\nabla}\hat{p} \cdot \hat{\mathbf{v}} d\hat{K}. \tag{3.11}$$

The following useful properties about Piola transformation can be found in [3] [15] [19].

$$\nabla \cdot \mathbf{v} = \frac{1}{J}\hat{\nabla} \cdot \hat{\mathbf{v}}, \qquad \mathbf{v} = \frac{1}{J}\mathcal{J}\hat{\mathbf{v}}. \tag{3.12}$$

$$\int_K \nabla \cdot \mathbf{v} dK = \int_{\hat{K}} \hat{\nabla} \cdot \hat{\mathbf{v}} d\hat{K}, \qquad \mathbf{v} = \frac{1}{J}\mathcal{J}\hat{\mathbf{v}}. \tag{3.13}$$

$$\int_{\partial K} \mathbf{v} \cdot \mathbf{n} ds = \int_{\partial \hat{K}} \hat{\mathbf{v}} \cdot \hat{\mathbf{n}} d\hat{s}, \qquad \mathbf{v} = \frac{1}{J}\mathcal{J}\hat{\mathbf{v}}. \tag{3.14}$$

Let the 2×2 positive definite matrices $\mathcal{K}$ and $\hat{\mathcal{K}}$ satisfy that

$$\hat{\mathcal{K}} = J\mathcal{J}^{-1}\mathcal{K}\mathcal{J}^{-t}, \tag{3.15}$$

then,

$$\mathcal{K} = \frac{1}{J}\mathcal{J}\hat{\mathcal{K}}\mathcal{J}^t, \quad \hat{\mathcal{K}}^{-1} = \frac{1}{J}\mathcal{J}^t\mathcal{K}^{-1}\mathcal{J} \quad \text{and} \quad \mathcal{K}^{-1} = J\mathcal{J}^{-t}\hat{\mathcal{K}}^{-1}\mathcal{J}^{-1}.$$

If

$$\hat{\mathbf{u}} = -\hat{\mathcal{K}}\hat{\nabla}\hat{p}, \tag{3.16}$$

then, (3.8), (3.9) and (3.15) yield

$$\mathbf{u} = \mathcal{P}_K\hat{\mathbf{u}} = -\mathcal{K}\nabla p. \tag{3.17}$$

Let us investigate the simple orthogonalities of Piola(under bilinear) transformation. The normal direction to the level curve, $\hat{x} = \hat{x}(x, y) =$ constant, in the xy-plane is

$$\mathbf{n}_{\hat{x}} = J\nabla\hat{x} = J(\hat{x}_x, \hat{x}_y)^t = (y_{\hat{y}}, -x_{\hat{y}})^t, \tag{3.18}$$

and the normal direction to the level curve, $\hat{y} = \hat{y}(x, y) =$ constant, in the xy-plane is

$$\mathbf{n}_{\hat{y}} = J\nabla\hat{y} = J(\hat{y}_x, \hat{y}_y)^t = (-y_{\hat{x}}, x_{\hat{x}})^t. \tag{3.19}$$

Similarly, the unit normals of the level curves, $\hat{x} =$ constant and $\hat{y} =$ constant in the $\hat{x}\hat{y}$-plane are

$$\hat{\mathbf{n}}_{\hat{x}} = \hat{\nabla}\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{n}}_{\hat{y}} = \hat{\nabla}\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.20}$$

respectively. According to (3.9), we have

$$\mathbf{n}_{\hat{x}} = J\mathcal{J}^{-t}\hat{\mathbf{n}}_{\hat{x}} \quad \text{and} \quad \mathbf{n}_{\hat{y}} = J\mathcal{J}^{-t}\hat{\mathbf{n}}_{\hat{y}}. \tag{3.21}$$

Let $\mathbf{u} = \mathcal{P}_K\hat{\mathbf{n}}_{\hat{x}}$ and $\mathbf{v} = \mathcal{P}_K\hat{\mathbf{n}}_{\hat{y}}$, then, at any point $(x, y) \in K$,

$$\mathbf{u} \cdot \mathbf{n}_{\hat{y}} = \hat{\mathbf{n}}_{\hat{x}} \cdot \hat{\mathbf{n}}_{\hat{y}} = 0, \quad \text{and} \quad \mathbf{v} \cdot \mathbf{n}_{\hat{x}} = \hat{\mathbf{n}}_{\hat{y}} \cdot \hat{\mathbf{n}}_{\hat{x}} = 0. \tag{3.22}$$

Let e_{i+1} be the line segment, $\overline{\mathbf{x}_i\mathbf{x}_{i+1}}$, and $|e_{i+1}| = |\mathbf{x}_{i,i+1}|$, be the length of $\overline{\mathbf{x}_i\mathbf{x}_{i+1}}$, $i = 1, 2, 3$. Let e_1 be the line segment, $\overline{\mathbf{x}_4\mathbf{x}_1}$, and $|e_1| = |\mathbf{x}_{41}|$. Then, the corresponding edges to e_i, $i = 1, 2, 3, 4$ of $\hat{K}$ are

$$\begin{aligned} \hat{e}_1 &= \{(\hat{x}, \hat{y}) \,|\, \hat{x} = 0, 0 \le \hat{y} \le 1\}, \quad \hat{e}_2 = \{(\hat{x}, \hat{y}) \,|\, \hat{y} = 0, 0 \le \hat{x} \le 1\}, \\ \hat{e}_3 &= \{(\hat{x}, \hat{y}) \,|\, \hat{x} = 1, 0 \le \hat{y} \le 1\}, \quad \hat{e}_4 = \{(\hat{x}, \hat{y}) \,|\, \hat{y} = 1, 0 \le \hat{x} \le 1\}. \end{aligned} \tag{3.23}$$

Let $\hat{\mathbf{n}}_i$ be the unit normal of $\hat{e}_i$, i.e., $\hat{\mathbf{n}}_i = (1, 0)^t$, $i = 1, 3$, and $\hat{\mathbf{n}}_i = (0, 1)^t$, $i = 2, 4$. Direct culculations of (2.18) and (2.19) lead to

$$|\mathbf{n}_{\hat{x}}|_{\hat{x}=0} = |e_1|, \quad |\mathbf{n}_{\hat{y}}|_{\hat{y}=0} = |e_2|, \quad |\mathbf{n}_{\hat{x}}|_{\hat{x}=1} = |e_3|, \quad |\mathbf{n}_{\hat{y}}|_{\hat{y}=1} = |e_4|, \tag{3.24}$$

then, the unit normal, $\mathbf{n}_i$, of e_i, $i = 1, 2, 3, 4$, are expressed by

$$\mathbf{n}_1 = \frac{\mathbf{n}_{\hat{x}}|_{\hat{x}=0}}{|e_1|}, \quad \mathbf{n}_2 = \frac{\mathbf{n}_{\hat{y}}|_{\hat{y}=0}}{|e_2|}, \quad \mathbf{n}_3 = \frac{\mathbf{n}_{\hat{x}}|_{\hat{x}=1}}{|e_3|}, \quad \mathbf{n}_4 = \frac{\mathbf{n}_{\hat{y}}|_{\hat{y}=1}}{|e_4|}. \tag{3.25}$$

Actually, $-\mathbf{n}_1$, $-\mathbf{n}_2$, $\mathbf{n}_3$, $\mathbf{n}_4$, are the unit outward normals of K. Definition (3.25) is convenient to define the unique nodal velocity values. Later, with an obuse of notation, we shall also use (3.25) to represent the unit noutward ormals of K and keep the sign differences in mind.

<u>Lemma 3.1.</u> *For the bilinear transformation* (3.2) *and the Piola transformation* (3.8), *if* $\mathbf{v} = \frac{1}{J}\mathcal{J}\hat{\mathbf{v}}$, *then,*

$$\int_{e_i} \mathbf{v} \cdot \mathbf{n}_i ds = \int_{\hat{e}_i} \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_i d\hat{s}, \qquad i = 1, 2, 3, 4. \tag{3.26}$$

<u>*Proof*</u> Note that by a change of variable on e_i, $ds = |e_i| d\hat{s}$, $i = 1, 2, 3, 4$. According to (3.18), (3.19), (3.21), (3.24) and (3.25),

$$|e_i| \mathbf{v} \cdot \mathbf{n}_i = \frac{1}{J}\mathcal{J}\hat{\mathbf{v}} \cdot (J\mathcal{J}^{-t}\hat{\mathbf{n}}_i) = \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_i, \quad i = 1, 2, 3, 4.$$

□

4. The Mixed Finite Element Spaces

In [19], page IX-21, if the elements are distorted rectangles, Thomas defines the discrete velocity space by imposing continuity in the normal direction at the mid-point of each element boundary edge with the same degree of polynomials as in the case of orthogonal rectangular elements. No stability or error estimates are discussed in [19] for this case. Actually, extra care has to be taken, otherwise, either the discrete velocity space may no longer be guaranteed a subspace of $\mathbf{V}$ or the divergence of the discrete velocity space is no longer a subspace of the discrete pressure space.

In this section we generate a pair of discrete velocity space, $\mathbf{V}_h$ and pressure space, W_h, such that

$$\begin{aligned} \mathbf{V}_h &\subset \mathbf{V} \subset H(\mathrm{div}; \Omega) \\ \mathrm{div}\mathbf{V}_h &\equiv W_h \subset W \subset L^2(\Omega), \end{aligned} \tag{4.1}$$

are satisfied. We shall keep the same number of unknowns as in [19], therefore, our mixed finite element method is in consistence with the lowest-order Raviart-Thomas orthogonal rectangular mixed finite element method [14][19]. That is, when the elements degenerate back to parallelograms, our mixed finite element method is exactly the lowest-order Raviart-Thomas orthogonal rectangular mixed finite element method. Most importantly, because of (4.1) our new mixed finite element

method satisfies the *B-B inf-sup* condition which is necessary and sufficient condition for existence and uniqueness (e.g., see [3] [13]). In §7, by means of our new mixed finite element method, we shall show the convergence, existence and uniqueness of Thomas's method.

THE PRESSURE SPACE W_h: Denote $\chi(K)$ the characteristic function of domain K, then, $W_h = \text{span}\{\chi(K)\}_{K\in\mathcal{T}_h}$. W_h is the piecewise constant space, then, $W_h \subset L^2(\Omega)$.

THE VELOCITY SPACE $\mathbf{V}_h$: In order to insure the normal direction continuity on the element edges, we require that the restriction of the nodal bases $\mathbf{b}_i \in \mathbf{V}_h$, $i = 1, 2, 3, 4$ on element K satisfy

$$\begin{aligned} \nabla \cdot \mathbf{b}_i &= \text{constant}, \\ \mathbf{b}_i \cdot \mathbf{n}_j|_{e_j} &= \delta_{ij}. \end{aligned} \tag{4.2}$$

By (3.12) and the orthogonality property of the Piola-bilinear transformation (3.22), (4.2) is equivalent to

$$\begin{aligned} \hat{\nabla} \cdot \hat{\mathbf{b}}_i &= J \cdot \text{constant}, \\ \hat{\mathbf{b}}_i \cdot \hat{\mathbf{n}}_j|_{\hat{e}_j} &= |e_i|\delta_{ij}. \end{aligned} \tag{4.3}$$

Note that J is a linear function of $\hat{x}$ and $\hat{y}$ by (3.6).

<u>LEMMA 4.1.</u> *There exists a $\hat{\mathbf{b}}_0 \in R^2$ defined on $\hat{K}$ such that*

$$\begin{aligned} \hat{\nabla} \cdot \hat{\mathbf{b}}_0 &= \frac{J}{|K|} - 1, \\ \hat{\mathbf{b}}_0 \cdot \hat{\mathbf{n}}_j|_{\hat{e}_j} &= 0, \quad j = 1, 2, 3, 4, \end{aligned} \tag{4.4}$$

and then,

$$\int_{\hat{K}} \hat{\nabla} \cdot \hat{\mathbf{b}}_0 d\hat{K} = 0. \tag{4.5}$$

<u>*Proof*</u> Indeed, let

$$\hat{\mathbf{b}}_0 = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}, \quad \zeta = \frac{\beta\hat{x}(\hat{x}-1)}{2|K|}, \quad \eta = \frac{\gamma\hat{y}(\hat{y}-1)}{2|K|}, \tag{4.6}$$

then, the $\hat{\mathbf{b}}_0$ given by (4.6) satisfies (4.4) and (4.5). In addition,

$$0 \le \max_{\hat{K}}\{|\zeta|, |\eta|\} < \frac{1}{4}. \tag{4.7}$$

□

¿From (4.6), we see that $\hat{\mathbf{b}}_0$ is solely determined by J. Now, let

$$\begin{aligned} \hat{\mathbf{b}}_1 &= |e_1| \begin{pmatrix} 1-\hat{x} \\ 0 \end{pmatrix} - |e_1|\hat{\mathbf{b}}_0, \quad \hat{\mathbf{b}}_3 = |e_3| \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix} + |e_3|\hat{\mathbf{b}}_0, \\ \hat{\mathbf{b}}_2 &= |e_2| \begin{pmatrix} 0 \\ 1-\hat{y} \end{pmatrix} - |e_2|\hat{\mathbf{b}}_0, \quad \hat{\mathbf{b}}_4 = |e_4| \begin{pmatrix} 0 \\ \hat{y} \end{pmatrix} + |e_4|\hat{\mathbf{b}}_0. \end{aligned} \tag{4.8}$$

Then, let

$$\mathbf{b}_i = \frac{1}{J}\mathcal{J}\hat{\mathbf{b}}_i, \qquad i = 1, 2, 3, 4. \tag{4.9}$$

It is straight forward to check that those nodal bases defined by (4.4) and (4.6) satisfy (4.3). Hence, the nodal bases given by (4.8) satisfy (4.2). Therefore, any $\mathbf{u} \in \mathbf{V}_h$, its restriction on K is uniquely determined by

$$\begin{aligned} \mathbf{u}|_K &= \sum_{i=1}^{4} u_i \mathbf{b}_i, \\ u_i &= \mathbf{u} \cdot \mathbf{n}_i|_{e_i}, \quad i = 1, 2, 3, 4. \end{aligned} \tag{4.10}$$

Clearly, $\mathrm{div}\mathbf{V}_h \subset W_h$, and from the dimension of $\mathbf{V}_h$ and W_h, the operator $\mathrm{div} : \mathbf{V}_h \to W_h$ is onto, therefore $W_h \subset \mathrm{div}\mathbf{V}_h$. Finally, the numerical space $\mathbf{V}_h \times W_h$ satisfies (4.1) as we desired.

<u>Remark 4.1.</u> From (4.6) and (4.7) we can see that the $\hat{x}$-components of $\hat{\mathbf{b}}_i$, $i = 1, 2, 3, 4$ are quadratic functions of $\hat{x}$ and the $\hat{y}$-components of $\hat{\mathbf{b}}_i$, $i = 1, 2, 3, 4$ are quadratic functions of $\hat{y}$. If the we drop the quadratic terms, the flux at all the boundary edges of $\hat{K}$ are the same, but unfortunately the divergence of the discrete velocity space will not be the piecewise constant pressure space any more. we shall overcome this problem in §7.

Now, we are ready to define the approximation problem for (2.4). The numerical scheme to approximate (2.4) is to seek a pair $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} (\mathcal{K}^{-1}\mathbf{u}_h, \mathbf{v}) &= (p_h, \nabla \cdot \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h, w) &= (f, w), \qquad w \in W_h. \end{aligned} \tag{4.11}$$

5. The Projections

For mixed finite element methods, projections play dominant roles in stability and error estimates, according to Fortin's Lemma [12]. First, the L^2-projection is defined in the usual sense, $Q_h : W \to W_h$, by

$$(f - Q_h f, w) = \int_\Omega (f - Q_h f) w d\Omega = 0, \qquad \forall\, f \in W,\ w \in W_h. \tag{5.1}$$

By definition (5.1), we can see that, if f is a piecewise constant function, $f \equiv Q_h f$, hence, applying Bramble-Hilbert Lemma [2] and the quasi-regularity (2.5), we have that

$$\|f - Q_h f\|_0 \le C \|f\|_1\, h. \tag{5.2}$$

The $H(\mathrm{div})$-projection is an extension [18] of the Raviart-Thomas projection [14][19], $\Pi_h : \mathbf{V} \to \mathbf{V}_h$, given by

$$(\nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u}), \nabla \cdot \mathbf{v}) = \int_\Omega \nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u}) \nabla \cdot \mathbf{v} d\Omega = 0, \qquad \forall\, \mathbf{u} \in \mathbf{V},\ \mathbf{v} \in \mathbf{V}_h. \tag{5.3}$$

If we take $\nabla \cdot \mathbf{v} = w \in W_h$ in (5.3) and combining (5.1), we derive that

$$\int_\Omega \nabla \cdot \Pi_h \mathbf{u}\, w\, d\Omega = \int_\Omega \nabla \cdot \mathbf{u}\, w\, d\Omega = \int_\Omega Q_h \nabla \cdot \mathbf{u}\, w\, d\Omega, \tag{5.4}$$

that is, $\mathrm{div}\Pi_h = Q_h \mathrm{div}$.

Now, we give the local expression for $\Pi_h \mathbf{u}$ explicitly for $\mathbf{u} \in (H^1(\Omega))^2$. Since $\Pi_h \mathbf{u} \in \mathbf{V}_h$, the restriction of $\Pi_h \mathbf{u}$ on K is given by

$$\Pi_h \mathbf{u}|_K = \sum_{i=1}^{4} u_i \mathbf{b}_i = \mathcal{P}_K \sum_{i=1}^{4} \frac{\hat{u}_i}{|e_i|} \hat{\mathbf{b}}_i \stackrel{\text{Def}}{=} \mathcal{P}_K \hat{\Pi}_h \hat{\mathbf{u}}|_{\hat{K}}. \tag{5.5}$$

In (5.3), taking $\mathbf{v} \in \mathbf{V}_h$ such that $\nabla \cdot \mathbf{v} = \chi(K) \in W_h$,

$$\begin{aligned} 0 &= \int_\Omega \nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u}) \cdot \chi(K) d\Omega = \int_K \nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u}) dK = \int_{\partial K} (\mathbf{u} - \Pi_h \mathbf{u}) \cdot \mathbf{n} ds \\ &= \int_{\hat{K}} \nabla \cdot (\hat{\mathbf{u}} - \hat{\Pi}_h \hat{\mathbf{u}}) d\hat{K} = \int_{\partial \hat{K}} (\hat{\mathbf{u}} - \hat{\Pi}_h \hat{\mathbf{u}}) \cdot \hat{\mathbf{n}} d\hat{s}. \end{aligned} \tag{5.6}$$

By Lemma 3.1., if we define $\hat{u}_i = \int_{\hat{e}_i} \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}_i d\hat{s}$, $i = 1, 2, 3, 4$, then,

$$u_i = \frac{1}{|e_i|} \int_{e_i} \mathbf{u} \cdot \mathbf{n}_i ds = \frac{\hat{u}_i}{|e_i|} = \frac{1}{|e_i|} \int_{\hat{e}_i} \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}_i d\hat{s}, \quad i = 1, 2, 3, 4, \tag{5.7}$$

therefore, $\Pi_h \mathbf{u}|_K = \mathcal{P}_K \hat{\Pi}_h \hat{\mathbf{u}}|_{\hat{K}}$ satisfy (5.6). Then, (5.5) can be rewritten as

$$\hat{\Pi}_h \hat{\mathbf{u}}|_{\hat{K}} = \begin{pmatrix} \hat{u}_1 + g_1 \hat{x} + g_0 \zeta \\ \hat{u}_2 + g_2 \hat{y} + g_0 \eta \end{pmatrix}, \tag{5.8}$$

where

$$\begin{aligned} g_1 &= \hat{u}_3 - \hat{u}_1 = \int_{\hat{K}} \partial_{\hat{x}} \hat{u}^{\hat{x}} d\hat{K}, \\ g_2 &= \hat{u}_4 - \hat{u}_2 = \int_{\hat{K}} \partial_{\hat{y}} \hat{u}^{\hat{y}} d\hat{K}, \\ g_0 &= \hat{u}_3 - \hat{u}_1 + \hat{u}_4 - \hat{u}_2 = \int_{\hat{K}} \hat{\nabla} \cdot \hat{\mathbf{u}} d\hat{K} = \int_K \nabla \cdot \mathbf{u} dK. \end{aligned} \tag{5.9}$$

¿From Cauchy-Schwarz inequality and (2.5),

$$|g_0| = |\int_K \nabla \cdot \mathbf{u} dK| \le \sqrt{|K|} \|\nabla \cdot \mathbf{u}\|_{0,K} \le C\, h \|\nabla \cdot \mathbf{u}\|_{0,K}. \tag{5.10}$$

We are ready to prove

<u>Lemma 5.1.</u> *Assume that* $\mathbf{u} \in (H^1(\Omega))^2$, *and* $\nabla \cdot \mathbf{u} \in H^1(\Omega)$, *then,*

$$\begin{aligned} \|\mathbf{u} - \Pi_h \mathbf{u}\|_0 &\le C \|\mathbf{u}\|_1\, h, \\ \|\nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u})\|_0 &\le C \|\nabla \cdot \mathbf{u}\|_1\, h. \end{aligned} \tag{5.11}$$

<u>Proof</u> The second estimate in (5.11) is actually (5.2). For the first one, we get

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{u} - \Pi_h \mathbf{u}\|_{0,K}^2 = \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{u} - \Pi_h \mathbf{u}) \cdot (\mathbf{u} - \Pi_h \mathbf{u}) dK,$$

and

$$\int_K (\mathbf{u} - \Pi_h \mathbf{u}) \cdot (\mathbf{u} - \Pi_h \mathbf{u}) dK = \int_{\hat{K}} \frac{1}{J} (\hat{\mathbf{u}} - \hat{\Pi}_h \hat{\mathbf{u}})^t \mathcal{J}^t \mathcal{J} (\hat{\mathbf{u}} - \hat{\Pi}_h \hat{\mathbf{u}}) d\hat{K}.$$

By the Bramble-Hilbert Lemma [2], it suffices to assume that each component of $\hat{\mathbf{u}}$ is at most linear, i.e,

$$\hat{\mathbf{u}} = \begin{pmatrix} \hat{u}_0^{\hat{x}} + (\hat{x} - \frac{1}{2})\partial_{\hat{x}} \hat{u}^{\hat{x}} + (\hat{y} - \frac{1}{2})\partial_{\hat{y}} \hat{u}^{\hat{x}} \\ \hat{u}_0^{\hat{y}} + (\hat{x} - \frac{1}{2})\partial_{\hat{x}} \hat{u}^{\hat{y}} + (\hat{y} - \frac{1}{2})\partial_{\hat{y}} \hat{u}^{\hat{y}} \end{pmatrix},$$

where $\hat{u}_0^{\hat{x}} = \hat{u}^{\hat{x}}(\frac{1}{2}, \frac{1}{2})$ and $\hat{u}_0^{\hat{y}} = \hat{u}^{\hat{y}}(\frac{1}{2}, \frac{1}{2})$. Then by (5.7),

$$\begin{aligned} \hat{u}_1 &= \hat{u}_0^{\hat{x}} - \frac{1}{2}\partial_{\hat{x}} \hat{u}^{\hat{x}}, \\ \hat{u}_2 &= \hat{u}_0^{\hat{y}} - \frac{1}{2}\partial_{\hat{y}} \hat{u}^{\hat{y}}. \end{aligned}$$

Denote

$$\hat{\mathbf{v}} = \hat{\mathbf{u}} - \hat{\Pi}_h \hat{\mathbf{u}} = \begin{pmatrix} (\hat{y} - \frac{1}{2})\partial_{\hat{y}} \hat{u}^{\hat{x}} - g_0 \zeta \\ (\hat{x} - \frac{1}{2})\partial_{\hat{x}} \hat{u}^{\hat{y}} - g_0 \eta \end{pmatrix}, \qquad \mathbf{v} = \mathcal{P}_K \hat{\mathbf{v}}.$$

According to [4] the local estimates for the bilinear map, the quasi-regularity assumption (2.5), (4.7) and (5.10), we have

$$\|\mathbf{v}\|_{0,K} \leq C\, h \|\mathbf{u}\|_{1,K}.$$

Summing over all the element $K \in \mathcal{T}_h$, we finish the proof of Lemma 5.1. □

Thus, if the domain Ω is a convex polygon, so that $H^2(\Omega)$-regularity is well defined, by Fortin's Lemma [12], the *B-B inf-sup* condition

$$\|w\|_0 \leq C \sup_{\mathbf{v} \in \mathbf{V}_h} \frac{(w, \nabla \cdot \mathbf{v})}{\|\mathbf{v}\|_{H(\mathrm{div})}}, \qquad \forall\, w \in W_h, \tag{5.12}$$

holds for $\mathbf{V}_h \times W_h$ defined in §4. Further, since the mass balance property is the property of $H(\mathrm{div}; \Omega)$, and, $\mathbf{V}_h \subset \mathbf{V} \subset H(\mathrm{div}; \Omega)$, the mass balance property is retained for $\mathbf{V}_h$ element-wise.

<u>Remark 5.1.</u> According to [12][20], $\mathrm{div}\mathbf{V}_h \subset W_h$ is not a necessary condition for (5.12) being held, which is also verified by the result of §7 in this paper.

6. Approximation Properties

Taking advantage of the well settled machinery for analyzing mixed schemes, by inequality (5.2) and Lemma 5.1., we have that

<u>Theorem 6.1.</u> *Under the assumption 2.1., let the pair* $(\mathbf{u}, p)$ *be the solution of* (2.4) *and* $(\mathbf{u}_h, p_h)$ *be the solution of* (4.6). *Then,*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_0 &\leq C\|\mathbf{u}\|_1\, h, \\ \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_0 &\leq C\|\nabla \cdot \mathbf{u}\|_1\, h, \\ \|p - p_h\|_0 &\leq C(\|p\|_1 + \|\mathbf{u}\|_1)\, h. \end{aligned} \tag{6.1}$$

Proof First, we show that

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le C\|\mathbf{u}-\Pi_h\mathbf{u}\|_0. \tag{6.2}$$

The error equations of (2.4) and (4.11) take the form of

$$\begin{array}{rcll} (\mathcal{K}^{-1}(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) & = & (p-p_h,\nabla\cdot\mathbf{v}), & \mathbf{v}\in\mathbf{V}_h, \\ (\nabla\cdot(\mathbf{u}-\mathbf{u}_h),w) & = & 0, & w\in W_h. \end{array} \tag{6.3}$$

Since $\mathcal{K}$ is bounded, (6.3) yields

$$\begin{aligned} \frac{1}{C}\|\mathbf{u}-\mathbf{u}_h\|_0^2 &\le (\mathcal{K}^{-1}(\mathbf{u}-\mathbf{u}_h),\mathbf{u}-\mathbf{u}_h) \\ &= (\mathcal{K}^{-1}(\mathbf{u}-\mathbf{u}_h),\mathbf{u}-\Pi_h\mathbf{u})+(\mathcal{K}^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi_h\mathbf{u}-\mathbf{u}_h) \\ &= (\mathcal{K}^{-1}(\mathbf{u}-\mathbf{u}_h),\mathbf{u}-\Pi_h\mathbf{u}). \end{aligned}$$

For $0=\nabla\cdot(\Pi_h\mathbf{u}-\mathbf{u}_h)\in W_h$, and the L^2-projection (5.1), it is straightforward to check that $(\mathcal{K}^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi_h\mathbf{u}-\mathbf{u}_h)=0$. Thus, (6.2) follows by applying Cauchy-Schwarz inequality, hence, Lemma 5.1 gives rise to the first inequality in (6.1). Again, because $0=\nabla\cdot(\Pi_h\mathbf{u}-\mathbf{u}_h)\in W_h$, (5.3) and (6.3) imply that

$$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0 \le \|\nabla\cdot(\mathbf{u}-\Pi_h\mathbf{u})\|_0. \tag{6.4}$$

Then, inequality (5.11) yields the second one of (6.1). Finally, applying the discrete *B-B inf-sup* condition (5.12),

$$\begin{aligned} \|Q_hp-p_h\|_0 &\le C\sup_{\mathbf{v}\in\tilde{\mathbf{V}}_h}\frac{(Q_hp-p_h,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_{H(\mathrm{div})}} \\ &= C\sup_{\mathbf{v}\in\tilde{\mathbf{V}}_h}\frac{(\mathcal{K}^{-1}(\mathbf{u}-\mathbf{u}_h),\mathbf{v})}{\|\mathbf{v}\|_{H(\mathrm{div})}} \le C\|\mathbf{u}-\mathbf{u}_h\|_0. \end{aligned}$$

Combining above results with (6.2), the proof is complete. □

By the duality argument of the standard mixed finite element methods(e.g., see [3]), we can derive the following superconvergence result for pressure alone

COROLLARY 6.2. *Under the assumption of Theorem 6.1., then,*

$$\|Q_hp-p_h\|_0 \le C(\|\mathbf{u}\|_1+\|\nabla\cdot\mathbf{u}\|_1)h^2. \tag{6.5}$$

7. On Thomas's Velocity Space

Denote by $\tilde{\mathbf{V}}_h$ the discrete velocity space on the same partition $\mathcal{T}_h$ defined by Thomas [19]. In this section, we overcome the difficulty of Thomas's velocity space: $\mathrm{div}\tilde{\mathbf{V}}_h$ is not a subspace of W_h and then, give theoretical support for the usage of Thomas's method. First, we set up some notations. From (4.8), if we define

$$\hat{\mathbf{t}}_i=\hat{\mathbf{b}}_i-|e_i|\hat{\mathbf{b}}_0,\quad i=1,2,\quad \hat{\mathbf{t}}_i=\hat{\mathbf{b}}_i+|e_i|\hat{\mathbf{b}}_0,\quad i=3,4, \tag{7.1}$$

then, for $\hat{\mathbf{b}}_0$ given by (4.6), let

$$\mathbf{b}_{0,K}=\mathcal{P}_K\hat{\mathbf{b}}_0,\quad \mathbf{t}_i=\mathcal{P}_K\hat{\mathbf{t}}_i,\quad i=1,2,3,4. \tag{7.2}$$

For any $\tilde{\mathbf{u}} \in \tilde{\mathbf{V}}_h$, its restriction on K is uniquely determined by

$$\begin{aligned} \tilde{\mathbf{u}}|_K &= \sum_{i=1}^{4} \tilde{u}_i \mathbf{t}_i, \\ \tilde{u}_i &= \tilde{\mathbf{u}} \cdot \mathbf{n}_i|_{e_i}, \quad i = 1,2,3,4. \end{aligned} \tag{7.3}$$

Since $\hat{K}$ is a unit square, the linear combination of $\hat{\mathbf{t}}_i$, $i = 1,2,3,4$ are tensor product of one-dimensional linear polynomials of $\hat{x}$ and $\hat{y}$ respectively. By definition [14][19], $\hat{\mathbf{t}}_i$, $i = 1,2,3,4$ are the bases for lowest-order Raviart-Thomas mixed finite orthogonal rectangular element bases on the reference element $\hat{K}$. But as long as $\mathcal{P}_K$ is not a constant matrix, $\mathbf{t}_i$, $i = 1,2,3,4$ do not satisfy (4.2). Therefore, div$\tilde{\mathbf{V}}_h \neq W_h$, even though $\tilde{\mathbf{V}}_h \subset \mathbf{V}$. The following lemma tells us how close the two spaces $\mathbf{V}_h$ and $\tilde{\mathbf{V}}_h$ are related.

<u>Lemma 7.1.</u> *For any $\mathbf{u} \in \mathbf{V}_h$, there is unique $\tilde{\mathbf{u}} \in \tilde{\mathbf{V}}_h$ such that $\mathbf{u}$ and $\tilde{\mathbf{u}}$ have exactly the same nodal values, i.e., on the boundaries of each element K,*

$$(\mathbf{u} - \tilde{\mathbf{u}}) \cdot \mathbf{n}_j = 0, \qquad i = 1,2,3,4, \qquad \forall\, K \in \mathcal{T}_h. \tag{7.4}$$

Further, if we let

$$\mathbf{u}^* = \mathbf{u} - \tilde{\mathbf{u}}, \tag{7.5}$$

then, for $\mathbf{u} \in \mathbf{V}_h$,

$$\begin{aligned} \|\mathbf{u} - \tilde{\mathbf{u}}\|_0 &= \|\mathbf{u}^*\|_0 \le C\|\nabla \cdot \mathbf{u}\|_0\, h, \\ \|\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})\|_0 &= \|\nabla \cdot \mathbf{u}^*\|_0 \le C\|\nabla \cdot \mathbf{u}\|_0\, h. \end{aligned} \tag{7.6}$$

<u>*Proof*</u> For $\mathbf{u} \in \mathbf{V}_h$, $u_i = \mathbf{u} \cdot \mathbf{n}_i|_{e_i}$ is a constant, then, combining with Lemma 3.1.,

$$u_i = \frac{1}{|e_i|} \int_{e_i} \mathbf{u} \cdot \mathbf{n}_i ds = \frac{1}{|e_i|} \int_{\hat{e}_i} \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}_i d\hat{s}.$$

Let

$$\hat{u}_i = |e_i| u_i,$$

then, by (5.8) and (5.9),

$$\begin{aligned} \mathbf{u} &= \sum_{K \in \mathcal{T}_h} \left(\sum_{i=1}^{4} u_i \mathbf{b}_i \right)_K = \sum_{K \in \mathcal{T}_h} \mathcal{P}_K \left(\sum_{i=1}^{4} \hat{u}_i \frac{\hat{\mathbf{b}}_i}{|e_i|} \right) \\ &= \sum_{K \in \mathcal{T}_h} \mathcal{P}_K \left(\sum_{i=1}^{4} \hat{u}_i \frac{\hat{\mathbf{t}}_i}{|e_i|} \right) + \sum_{K \in \mathcal{T}_h} \left(\int_K \nabla \cdot \mathbf{u} dK \right) \mathbf{b}_{0,K} \\ &\overset{\text{Def}}{=} \tilde{\mathbf{u}} + \mathbf{u}^*. \end{aligned} \tag{7.7}$$

By definition (4.10), (7.3) and the nodal bases relation (7.1), the first assertion (7.4) is proved. Now, from (2.5), (3.2), (3.4) and (4.7), we have

$$|K||\mathbf{b}_0| = \sqrt{\mathbf{b}_0 \cdot \mathbf{b}_0} \le \frac{|K|}{4J} \sqrt{(x_{\hat{x}} + x_{\hat{y}})^2 + (y_{\hat{x}} + y_{\hat{y}})^2} \le Ch.$$

Then,

$$\begin{aligned}\|\mathbf{u}^*\|_0^2 &= \sum_{K\in\mathcal{T}_h}\left(\int_K \nabla\cdot\mathbf{u}dK\right)^2\int_K \mathbf{b}_0\cdot\mathbf{b}_0 dK\\ &\le Ch^2\sum_{K\in\mathcal{T}_h}\|\nabla\cdot\mathbf{u}\|_{0,K}^2 = C\|\nabla\cdot\mathbf{u}\|_0^2\,h^2.\end{aligned}$$

For (7.6), from (7.7),

$$\nabla\cdot\mathbf{u}^*|_K = \left(\int_K \nabla\cdot\mathbf{u}dK\right)\nabla\cdot\mathbf{b}_0,$$

and from (3.12),

$$\nabla\cdot\mathbf{b}_0 = \frac{1}{J}\hat{\nabla}\cdot\hat{\mathbf{b}}_0 = \frac{1}{|K|}\left[1-\frac{|K|}{J}\right].$$

Thus,

$$|K|^2\int_K(\nabla\cdot\mathbf{b}_0)^2dK = \int_{\hat{K}}\frac{(J-|K|)^2}{J}d\hat{K} \le \frac{(\beta^2+\gamma^2)}{12J_*} \le Ch^2,$$

for

$$\begin{aligned}\int_{\hat{K}}(J-|K|)^2d\hat{K} &= \int_0^1\int_0^1\left[\beta^2(\hat{x}-\frac{1}{2})^2+\gamma^2(\hat{y}-\frac{1}{2})^2-2\beta\gamma(\hat{x}-\frac{1}{2})(\hat{y}-\frac{1}{2})\right]d\hat{x}d\hat{y}\\ &= \frac{(\beta^2+\gamma^2)}{12},\end{aligned}$$

$0 < J_* = \min_{\hat{K}}\{J\} \le Ch^2$ and $0 \le |\beta|,\ |\gamma| \le Ch^2$. Finally,

$$\begin{aligned}\|\nabla\cdot\mathbf{u}^*\|_0^2 &= \sum_{K\in\mathcal{T}_h}(\int_K\nabla\cdot\mathbf{u}dK)^2\int_K(\nabla\cdot\mathbf{b}_0)^2dK\\ &\le Ch^2\sum_{K\in\mathcal{T}_h}\|\nabla\cdot\mathbf{u}\|_{0,K}^2 = C\|\nabla\cdot\mathbf{u}\|_0^2\,h^2.\end{aligned}$$

□

Apparently, Lemma 7.1. also implies that $\tilde{\mathbf{V}}_h \subset \mathbf{V}$. Now, we define another approximation problem for (2.4). The numerical scheme to approximate (2.4) is to seek a pair $(\tilde{\mathbf{u}}_h, p_h) \in \tilde{\mathbf{V}}_h \times W_h$ such that

$$\begin{aligned}(\mathcal{K}^{-1}\tilde{\mathbf{u}}_h,\tilde{\mathbf{v}}) &= (\tilde{p}_h,\nabla\cdot\mathbf{v}), & \tilde{\mathbf{v}}\in\tilde{\mathbf{V}}_h,\\ (\nabla\cdot\tilde{\mathbf{u}}_h, w) &= (f,w), & w\in W_h.\end{aligned} \tag{7.8}$$

Combined with inequality (5.2) and Lemma 5.1., Lemma 7.1 enables us to show that

<u>Theorem 7.2.</u> *Under the assumption 2.1., let the pair* $(\mathbf{u},p)$ *be the solution of* (2.4) *and* $(\tilde{\mathbf{u}}_h,\tilde{p}_h)$ *be the solution of* (7.8). *In addition, if* h *is sufficiently small, then,*

$$\begin{aligned}\|p-\tilde{p}_h\|_0 &\le C(\|p\|_1+\|\mathbf{u}\|_1)\,h,\\ \|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0 &\le C\|\mathbf{u}\|_1\,h,\\ \|\nabla\cdot(\mathbf{u}-\tilde{\mathbf{u}}_h)\|_0 &\le C\|\nabla\cdot\mathbf{u}\|_1\,h.\end{aligned} \tag{7.9}$$

<u>*Proof*</u> The error equations of (2.4) and (7.8) take the form of

$$\begin{array}{rcll} (\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h,\tilde{\mathbf{v}}) & = & (p-\tilde{p}_h,\nabla\cdot\tilde{\mathbf{v}}), & \tilde{\mathbf{v}}\in\tilde{\mathbf{V}}_h, \\ (\nabla\cdot(\mathbf{u}-\tilde{\mathbf{u}}_h),w) & = & 0, & w\in W_h. \end{array} \tag{7.10}$$

Since $\mathcal{K}$ is bounded, (7.10) yields

$$\begin{array}{rcl} \frac{1}{C}\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0^2 & \le & (\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\mathbf{u}-\tilde{\mathbf{u}}_h) \\ & = & (\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\mathbf{u}-\Pi_h\mathbf{u})+(\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\Pi_h\mathbf{u}-\tilde{\mathbf{u}}_h). \end{array} \tag{7.11}$$

Let $\mathbf{u}^*$ be the $O(h)$-perturbation part of $\Pi_h\mathbf{u}$ such that $\Pi_h\mathbf{u}-\mathbf{u}^*\in\tilde{\mathbf{V}}_h$. Similarly, let $\mathbf{u}_h^*$ be the $O(h)$-perturbation part of $\tilde{\mathbf{u}}_h$ such that $\tilde{\mathbf{u}}_h+\mathbf{u}_h^*\in\mathbf{V}_h$. Thus, Lemma 7.1. gives rise to $\|\mathbf{u}^*\|_0\le Ch\|\nabla\cdot\mathbf{u}\|_0$ and $\|\mathbf{u}_h^*\|_0\le Ch\|\nabla\cdot\mathbf{u}\|_0$. Rewrite the second term in (7.11) as

$$(\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\Pi_h\mathbf{u}-\tilde{\mathbf{u}}_h)=(\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\Pi_h\mathbf{u}-\mathbf{u}^*-\tilde{\mathbf{u}}_h)+(\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\mathbf{u}^*).$$

Since

$$|(\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\mathbf{u}^*)|\le C\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0\|\mathbf{u}^*\|_0\le C\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0\|\nabla\cdot\mathbf{u}\|_0\,h,$$

and $\Pi_h\mathbf{u}-\mathbf{u}^*-\tilde{\mathbf{u}}_h\in\tilde{\mathbf{V}}_h$, the first equation of (7.10) gives rise to

$$(\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\Pi_h\mathbf{u}-\mathbf{u}^*-\tilde{\mathbf{u}}_h)=(p-\tilde{p}_h,\nabla\cdot(\Pi_h\mathbf{u}-\mathbf{u}^*-\tilde{\mathbf{u}}_h)).$$

Then, (5.4) and the second equation of (7.10) lead to

$$\begin{array}{rcl} |(p-\tilde{p}_h,\nabla\cdot(\Pi_h\mathbf{u}-\mathbf{u}^*-\tilde{\mathbf{u}}_h))| & = & |(p-\tilde{p}_h,\nabla\cdot(\mathbf{u}_h^*-\mathbf{u}^*))+(Q_hp-\tilde{p}_h,\nabla\cdot\mathbf{u}^*)| \\ & \le & C\left(h\|p\|_1+\|Q_hp-\tilde{p}_h\|_0\right)\|\nabla\cdot\mathbf{u}\|_0\,h. \end{array}$$

Put the above results in (7.11), we derive that

$$\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0^2\le C\left(\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0\|\mathbf{u}\|_1+\|Q_hp-\tilde{p}_h\|_0\|\nabla\cdot\mathbf{u}\|_0+h\|p\|_1\|\nabla\cdot\mathbf{u}\|_0\right)h. \tag{7.12}$$

Next, we show that, if h is sufficiently small,

$$\|Q_hp-\tilde{p}_h\|_0\le C\left(\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0+h^2\|p\|_1\right). \tag{7.13}$$

If $\mathbf{v}\in\mathbf{V}_h$ is arbitrary, then, Lemma 7.1. implies that $\mathbf{v}=\tilde{\mathbf{v}}+\mathbf{v}^*$, $\tilde{\mathbf{v}}\in\tilde{\mathbf{V}}_h$ such that $\|\mathbf{v}^*\|_0\le Ch\|\nabla\cdot\mathbf{v}\|_0$ and $\|\nabla\cdot\mathbf{v}^*\|_0\le Ch\|\nabla\cdot\mathbf{v}\|_0$. Since $\mathbf{v}\in\mathbf{V}_h$, (5.1) and the first equation of (7.10) yield

$$\begin{array}{rcl} (Q_hp-\tilde{p}_h,\nabla\cdot\mathbf{v}) & = & (p-\tilde{p}_h,\nabla\cdot\mathbf{v}) \\ & = & (p-\tilde{p}_h,\nabla\cdot\tilde{\mathbf{v}})+(p-\tilde{p}_h,\nabla\cdot\mathbf{v}^*) \\ & = & (\mathcal{K}^{-1}(\mathbf{u}-\tilde{\mathbf{u}}_h),\tilde{\mathbf{v}})+(p-\tilde{p}_h,\nabla\cdot\mathbf{v}^*). \end{array} \tag{7.14}$$

Triangal inequality and Lemma 7.1. yield

$$\|\tilde{\mathbf{v}}\|_0\le\|\mathbf{v}\|_0+Ch\|\nabla\cdot\mathbf{v}\|_0\le C\|\mathbf{v}\|_{H(\mathrm{div})},$$

and

$$\|\nabla\cdot\mathbf{v}^*\|_0\le C\|\nabla\cdot\mathbf{v}\|_0\,h\le C\|\mathbf{v}\|_{H(\mathrm{div})}\,h.$$

Applying Cauchy-Schwarz inequality to (7.14), we have

$$|(Q_h p - \tilde{p}_h, \nabla \cdot \mathbf{v})| \leq C \left(\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0 + h^2 \|p\|_1 + h \|Q_h p - \tilde{p}_h\|_0 \right) \|\mathbf{v}\|_{H(\mathrm{div})}. \tag{7.15}$$

Then, (5.12) and (7.15) give rise to

$$\|Q_h p - \tilde{p}_h\|_0 \leq C \left(\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0 + h^2 \|p\|_1 + h \|Q_h p - \tilde{p}_h\|_0 \right),$$

and if h is sufficiently small, (7.13) holds. Again, if h is sufficiently small, (5.2), (5.11), (7.12) and (7.13) yield the first two inequalities of (7.9). Finally, by (5.11) and the second equation in (7.10),

$$\begin{aligned}
\|\nabla \cdot \mathbf{u} - \tilde{\mathbf{u}}_h\|_0^2 &= (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h), \nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u})) + (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h), \nabla \cdot (\Pi_h \mathbf{u} - \tilde{\mathbf{u}}_h)) \\
&= (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h), \nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u})) + (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h), \nabla \cdot \tilde{\mathbf{u}}_h^*) \\
&\quad + (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h), \nabla \cdot (\Pi_h \mathbf{u} - \tilde{\mathbf{u}}_h - \tilde{\mathbf{u}}_h^*)) \\
&= (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h), \nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u})) + (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h), \nabla \cdot \tilde{\mathbf{u}}_h^*) \\
&\leq C \|\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}_h)\|_0 \|\nabla \cdot \mathbf{u}\|_1 \, h.
\end{aligned}$$

Thus, Theorem 7.2. is proved.

□

<u>Remark 7.1.</u> Throughout the argument of Theorem 7.1., we have tacitly assumed that (7.8) is solvable. The existence and uniqueness of (7.8) can be established from (7.12) and (7.13). Since for finite dimensional cases, uniqueness implies existence, we only need to demonstrate uniqueness. Momentarily, we interpret $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ in (7.8) as a solution pair of the homogeneous problem. If h is sufficiently small, (7.13) yields

$$\|\tilde{p}_h\|_0 \leq C \|\tilde{\mathbf{u}}_h\|_0,$$

and (7.12), with $f = 0$, shows that

$$\|\tilde{\mathbf{u}}_h\|_0 = 0.$$

Therefore, $\tilde{p}_h$ and $\tilde{\mathbf{u}}_h$ vanish for small h. Finally, the necessity of the discrete *B-B inf-sup* condition for existence and uniqueness (e.g., see [13]) implies that the discrete *B-B inf-sup* condition holds for $\tilde{\mathbf{V}}_h \times W_h$ also, if h is sufficiently small.

Similar to Corollary 6.2., we have the superconvergence result for pressure alone

<u>Corollary 7.3.</u> *Under the assumption of Theorem 7.2., then,*

$$\|Q_h p - \tilde{p}_h\|_0 \leq C(\|\mathbf{u}\|_1 + \|\nabla \cdot \mathbf{u}\|_1) h^2. \tag{7.16}$$

<u>*Proof*</u> Define the adjoint problem of (1.2)

$$\begin{aligned}
\mathcal{K}^{-1}\mathbf{q} &= -\nabla \phi, && \text{in} \quad \Omega, \\
\nabla \cdot \mathbf{q} &= Q_h p - \tilde{p}_h, && \text{in} \quad \Omega, \\
\phi &= 0, && \text{on} \quad \Gamma_1, \\
\mathbf{q} \cdot \mathbf{n} &= 0, && \text{on} \quad \Gamma_2,
\end{aligned}$$

and assume that $\|\phi\|_1,\ \|\mathbf{q}\|_1 \le C\|Q_h p - \tilde{p}_h\|_0$. Let $\mathbf{q}^*$ be the $O(h)$-perturbation part of $\Pi_h\mathbf{q}$ such that $\Pi_h\mathbf{q} - \mathbf{q}^* \in \tilde{\mathbf{V}}_h$ and $\|\mathbf{q}^*\|_0 \le Ch\|\nabla\cdot\mathbf{q}\|_0$ by Lemma 7.1. Then,

$$\begin{aligned}
\|Q_h p - \tilde{p}_h\|_0^2 &= (Q_h p - \tilde{p}_h, \nabla\cdot\mathbf{q}) = (Q_h p - \tilde{p}_h, \nabla\cdot\Pi_h\mathbf{q}) = (p - \tilde{p}_h, \nabla\cdot\Pi_h\mathbf{q}) \\
&= (p - \tilde{p}_h, \nabla\cdot(\Pi_h\mathbf{q} - \mathbf{q}^*)) + (p - \tilde{p}_h, \nabla\cdot\mathbf{q}^*) \\
&= (\mathcal{K}^{-1}(\mathbf{u} - \tilde{\mathbf{u}}_h), \Pi_h\mathbf{q} - \mathbf{q}^*) + (p - \tilde{p}_h, \nabla\cdot\mathbf{q}^*) \\
&= (\mathcal{K}^{-1}(\mathbf{u} - \tilde{\mathbf{u}}_h), \Pi_h\mathbf{q} - \mathbf{q}) + (\mathcal{K}^{-1}\mathbf{q}, \mathbf{u} - \tilde{\mathbf{u}}_h) \\
&\quad - (\mathcal{K}^{-1}(\mathbf{u} - \tilde{\mathbf{u}}_h), \mathbf{q}^*) + (p - \tilde{p}_h, \nabla\cdot\mathbf{q}^*) \\
&= (\mathcal{K}^{-1}(\mathbf{u} - \tilde{\mathbf{u}}_h), \Pi_h\mathbf{q} - \mathbf{q}) + (\phi, \nabla\cdot(\mathbf{u} - \tilde{\mathbf{u}}_h)) \\
&\quad - (\mathcal{K}^{-1}(\mathbf{u} - \tilde{\mathbf{u}}_h), \mathbf{q}^*) + (p - \tilde{p}_h, \nabla\cdot\mathbf{q}^*) \\
&= (\mathcal{K}^{-1}(\mathbf{u} - \tilde{\mathbf{u}}_h), \Pi_h\mathbf{q} - \mathbf{q}) + (\phi - Q_h\phi, \nabla\cdot(\mathbf{u} - \tilde{\mathbf{u}}_h)) \\
&\quad - (\mathcal{K}^{-1}(\mathbf{u} - \tilde{\mathbf{u}}_h), \mathbf{q}^*) + (p - \tilde{p}_h, \nabla\cdot\mathbf{q}^*) \\
&\le C\left(\|\mathbf{u}\|_1\|\mathbf{q}\|_1 + \|\nabla\cdot\mathbf{u}\|_1\|\phi\|_1 + (\|\mathbf{u}\|_1 + \|p\|_1\|\nabla\cdot\mathbf{q}\|_0\right)h^2 \\
&\quad + C_1\|Q_h p - \tilde{p}_h\|_0\|\nabla\cdot\mathbf{q}\|_0\, h.
\end{aligned}$$

If h is sufficiently small, then, (7.16) is valid.

□

8. Conclusion Remarks

¿From Theorem 6.1. and Theorem 7.2., we can see that these two mixed method discretizations (4.11) and (7.8) provide $O(h)$-order accurace for both pressure and velocity. The only shortcoming of (7.8) is requiring h be sufficiently small. This provides theoretical support for the computational results of Farmer et. al. [11] and partially for Russell's (not the superconvergence for velocity) [16]. The discrete space $\mathbf{V}_h \times W_h$ given in §4, not only serves as the theoretical foundation for both methods but also offers a reasonable solver to approximate (2.4). In computational implementations, compared with the space $\tilde{\mathbf{V}}_h \times W_h$ given in §7, the only draw back of method (4.11) is the small amount of work to calculate the integrals of the quadratic terms element-wise in the mass matrix formulation. The mass matrices of these two methods have exactly the same band structure.

These two methods would be superior in accuracy, to the corresponding triangular method, if some superconvergence results for velocity can be derived, similar to [6][7][8][9] for orthogonal grid. But this is beyond the purpose of this paper.

Finally, the piecewise constant Lagrange multipliers on the boundaries of elements are well defined for our mixed finite element space defined in §4 and the Thomas's [19] given in §7. Therefore, the superconvergence results of the Lagrange multipliers will follow without any difficulties. The great advantage of introducing Lagrange multipliers is that we can convert the saddle-point problem (4.11) and (7.8) to positive definite algebraic systems. We refer to Arnold and Brezzi [1] for details on the orthogonal rectangular grids.

References

[1] D. N. Arnold and F. Brezzi, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, R.A.I.R.O. Anal. Numer. 19(1985).

[2] J. H. Bramble and S. R. Hilbert, *Bounds for a class of linear functionals with application to Hermite interpolation*, Numer. Math., 16 (1971), pp. 362–369.

[3] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, 1991.

[4] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[5] R. E. Ewing, R. D. Lazarov, and J. Shen, *Mixed finite element methods on logically cubical grids*, (In preparation).

[6] R. E. Ewing, R. D. Lazarov, and J. Wang, *Superconvergence of the velocity along the Gauss lines in mixed finite element methods*, SIAM J. Numer. Anal., 28 (1991), pp. 1051–1029.

[7] R. E. Ewing, R. C. Sharpley and J. Shen, *On accuracy of the lowest-order quadrilateral mixed finite element discretization in solving heterogeneous flow problems*, (In preparation).

[8] R. E. Ewing and J. Shen, *A discretization scheme and error estimate for second-order elliptic problems with discontinuous coefficients*, preprint of ISC, TAMU 1993.

[9] R. E. Ewing, J. Shen and J. Wang, *Point distributed algorithms for second-order elliptic equations in mixed form*, Proceedings of Fifth Korean Advanced Institute for Science Technology International Workshop in Analysis, U. J. Choi, et. al. eds. Taejon, Korea, 1993, pp. 53–76.

[10] R. S. Falk and J. E. Osborn, *Remarks on mixed finite element methods for problems with rough coefficients*, Math. Comp., vol. 62(205), 1994, pp. 1–19.

[11] C. L. Farmer, D. E. Heath and D. O. Moody, *A global optimization approach to grid generation*, SPE 21236, Proceedings of 11th SPE Symposium on Reservoir Simulation, Anaheim, CA, Feb. 17-20, 1991.

[12] M. Fortin, *Analysis of the convergence of mixed finite element methods*, R.A.I.R.O. Anal. Numer., 11 (1977), pp. 341–354.

[13] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, 1986.

[14] P. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, Mathematics Aspects of the Finite Element Method, Lecture Notes in Mathematics 606, Springer-Verlag (1977) pp. 292–315.

[15] J. E. Robert and J. M. Thomas, Mixed and hybrid methods, in *Handbook of Numerical Analysis*, (P.G. Ciarlet and J.L. Lions, eds.), *Vol. II*, *Finite Element Methods (Part 1)*, North-Holland, Amsterdam (1889).

[16] T. F. Russell, *Rigorous discretizations for irregular block-centered grids*, Colloquium in the Institute for Scientific Computation, Texas A & M University, Texas, April 1994.

[17] J. Shen, *Mixed Finite Element Methods: Analysis and Computational Aspects*, Ph.D. Thesis, University of Wyoming, 1992.

[18] J. Shen, *A block finite difference scheme for second-order elliptic problems with discontinuous coefficients*, SIAM J. Numer. Anal. 33-2 (to appear).

[19] J. M. Thomas, *Sur l'analyse numérique des méthods d'éléments finis hybrides et mixtes*, Thèse d'Etat, Université Pierre et Marie Curie, Paris (1977).

[20] Personal discussion wiht Prof. J. M. Thomas.

Department of Mathematics and
Institute for Scientific Computation
Texas A & M University
College Station, TX 77843-3404
E-mail : shen@isc.tamu.edu