Superconvergence of mixed finite element methods for parabolic problems with nonsmooth initial data

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Abstract

Semidiscrete mixed finite element approximation to parabolic initial-boundary value problems is introduced and analyzed. Superconvergence estimates for both pressure and velocity are obtained. The estimates for the errors in pressure and velocity depend on the smoothness of the initial data including the limiting cases of data in L^2 and data in H^r, for r sufficiently large. Because of the smoothing properties of the parabolic operator these estimates for large time levels essentailly coincide with the estimates obtained earlier for smooth solutions. However, for small time intervals we obtain the correct convergence orders for nonsmooth data.

Keywords Parabolic problem, mixed finite element method, superconvergence
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1 Introduction

Since the pioneering work of Raviart and Thomas [22] the mixed finite element approximations to second order elliptic problems have drawn the attention of many specialists on numerical partial differential equations. This method provides direct approximation of the physical quantities such as fluxes or velocities and leads to schemes that are locally conservative. However, this approach leads to saddle point problems that are more difficult to approximate and solve. Due largely to Babuska [1] and Brezzi [3], it is now well understood that the finite element spaces approximating different physical quantities (pressure and velocity, temperature and flux, etc.) cannot be chosen independently. Then the so-called *inf-sup* condition of Babuska–Brezzi is essential if one wants to construct unconditionally stable schemes with optimal convergence rates.

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Superconvergence results are important from an application point of view since under reasonable assumptions on the grid and with additional smoothness of the solution, they provide higher order accuracy. To our knowledge the first superconvergence results in the mixed method for second order elliptic problems were proven by Douglas and Milner in [9] and by Nakata, Weiser and Wheeler in [20]. Since then a wide variety of results has been obtained by many other authors, see, e.g., Duran [12], Douglas and Wang [11], Wang [28], and Ewing, Lazarov and Wang [14].

The error analysis of mixed finite element methods applied to time-dependent problems was developed by Johnson and Thomée in [18]. Then, Squeff in [24] using the quasi-projections of Douglas, Dupont and Wheeler [8], and the averaging method of Bramble and Schatz [2] obtained superconvergence rates for parabolic mixed finite element solutions in R^2. Recently, using a different approach, Ewing and Lazarov, in [13], established a superconvergence analysis at Gauss lines when using rectangular Raviart-Thomas finite elements of index $r \geq 0$.

The superconvergence results of Squeff [24] and Ewing and Lazarov [13] are valid for problems that have sufficiently smooth solutions uniformly in time. However, many practical problems involve irregularities, such as, in the initial data. This may results in a break-down of the uniform regularity of the solutions, which makes these estimates not applicable to such problems. Fortunately, the linear parabolic operators have the so-called *smoothing property*. Namely, even for nonsmooth initial data given at $t = 0$ or initial data that is not compatible with the boundary condition, the solution of the homogeneous parabolic equation is sufficiently smooth away from $t = 0$. However, the same solution has a singularity of the form $t^{-\beta}$ with some β positive. This kind of smoothing property has been established also for the standard Galerkin parabolic finite element equations and used to derive optimal error estimates for problems with rough initial data, see, e.g., Luskin and Rannacher [19], Schatz, Thomée and Wahlbin [23], Thomée [25] and Rannacher [21]. Furthermore, superconvergence results for the gradient for the standard Galerkin finite element methods with initial data of low regularity were obtained by Thomée, Xu and Zhang in [26] and by Chen in [6].

The main goal of this paper is to prove convergence and superconvergence error estimates for the mixed finite element methods when applied to parabolic problems with rough initial data. These estimates are quite similar to those obtained previously for conventional Galerkin finite element methods. Such estimates have been derived for the lowest order mixed finite elements by Johnson and Thomée in [18]. Our approach is different from the method of Johnson and Thomée [18]; the proofs used in this paper are based on the energy method and on the parabolic duality argument.

Let $\Omega \in R^2$ be a smooth and bounded domain. We consider the following parabolic problem:

$$\begin{cases} p_t - Ap = f, & \text{in } \Omega \times (0, \infty), \\ p = 0, & \text{on } \partial\Omega \times (0, \infty), \\ p|_{t=0} \text{ is given function in } L^2(\Omega), & \end{cases} \tag{1}$$

where $A = \nabla \cdot (a\nabla p)$ and $a = a(x,t)$ is a sufficiently smooth function that is bounded below by a positive constant on $\Omega \times (0, \infty)$. Our purpose is to solve problem (1) using mixed finite element methods. To describe the mixed variational form for (1), as usual, we introduce two Hilbert spaces. Let

$$W = L^2(\Omega), \qquad \boldsymbol{V} = \left\{ \boldsymbol{\varphi} \in L^2(\Omega)^2,\ \nabla \cdot \boldsymbol{\varphi} \in L^2(\Omega) \right\},$$

and let the space $\boldsymbol{V}$ be equipped with the norm $\| \boldsymbol{\varphi} \|_{\boldsymbol{V}} = (\| \boldsymbol{\varphi} \|^2 + \| \nabla \cdot \boldsymbol{\varphi} \|^2)^{1/2}$. The inner product and the norm in $L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. For the sake of simplicity, $(\cdot,\cdot)$ and $\|\cdot\|$ are also used as the inner product and norm, respectively, in the product space $L^2(\Omega)^2$.

Throughout this paper $H^i(\Omega)$ denotes the standard Sobolev spaces $W^{i,2}(\Omega)$, with $H^0(\Omega) = L^2(\Omega)$ and

$$H_0^1(\Omega) = \{ v \in H^1(\Omega),\ v = 0 \text{in } \partial\Omega \}.$$

We also denote by $\boldsymbol{H}^i(\Omega)$ the vector analog of $H^i(\Omega)$. For $k > 0$, the negative norm $\| v \|_{H^{-k}(\Omega)}$ is defined by

$$\| v \|_{H^{-k}(\Omega)} = \sup_{\varphi \in C_0^\infty(\Omega)} \frac{(v, \varphi)}{\| \varphi \|_{H^k(\Omega)}}.$$

In order to define the compatibility conditions for smooth solutions of initial value problems for the parabolic equation (1) we need the space $\dot{H}_t^j(\Omega)$, $0 \le t < \infty$, (see, e.g., [17], [25]):

$$\dot{H}_t^k(\Omega) = \{ v_0 \in H^k(\Omega) :\ v_j = -\sum_{l=0}^{j-1} C_l^{j-1} A^{j-1-l}(t) v_l \in H_0^1(\Omega), j < k/2 \}.$$

Further, let $\dot{H}^k(\Omega) = \dot{H}_0^k(\Omega)$. Associated with $\dot{H}_t^k(\Omega)$, the norm in $\dot{H}_t^{-k}(\Omega)$ is defined by

$$\| v \|_{\dot{H}_t^{-k}(\Omega)} = \sup_{\varphi \in \dot{H}_t^k(\Omega)} \frac{(v, \varphi)}{\| \varphi \|_{H^k(\Omega)}}.$$

By introducing the velocity $\boldsymbol{u} = a\nabla p$, the problem (1) is equivalent to finding $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ such taht

$$\begin{cases} (p_t, \psi) - (\nabla \cdot \boldsymbol{u}, \psi) = (f, \psi), & \forall\ \psi \in W, t \in (0, \infty), \\ (\alpha \boldsymbol{u}, \boldsymbol{\varphi}) + (p, \nabla \cdot \boldsymbol{\varphi}) = 0, & \forall\ \boldsymbol{\varphi} \in \boldsymbol{V}, t \in (0, \infty), \\ p(0) \in L^2(\Omega) \text{ is given }. \end{cases} \tag{2}$$

Here $p_t = \partial p / \partial t$ and $\alpha = a^{-1}$. We note that the boundary condition $p = 0$ on $\partial\Omega$ is implicitly contained in (2).

Given the finite-dimensional spaces $W_h \subset W$ and $\boldsymbol{V}_h \subset \boldsymbol{V}$, $0 < h < 1$, the so-called mixed finite element approximation $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ of $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ is the solution of the following problem:

$$\begin{cases} (p_{h,t}, \psi_h) - (\nabla \cdot \boldsymbol{u}_h, \psi_h) = (f, \psi_h), & \forall\ \psi_h \in W_h, t \in (0,\infty), \\ (\alpha \boldsymbol{u}_h, \boldsymbol{\varphi}_h) + (\nabla \cdot \boldsymbol{\varphi}_h, p_h) = 0, & \forall\ \boldsymbol{\varphi}_h \in \boldsymbol{V}_h, t \in (0,\infty), \\ p_h(0) \in W_h \text{ is given.} \end{cases} \tag{3}$$

We note that $\boldsymbol{u}_h(0)$ is determined by $p_h(0)$ through the second equation of (3).

To ensure the existence and convergence of the solution of the above formulation, we assume that

$$\nabla \cdot \boldsymbol{V}_h \subset W_h$$

and there exists a linear operator $\boldsymbol{\Pi}_h$: $\boldsymbol{V} \to \boldsymbol{V}_h$ such that

$$\nabla \cdot \boldsymbol{\Pi}_h = Q_h \nabla \cdot . \tag{4}$$

Here, the operator $Q_h :\ W \to W_h$ is the L^2-projection, i.e.,

$$(\psi - Q_h\psi, \psi_h) = 0, \quad \forall\ \psi \in W, \psi_h \in W_h.$$

The identity (4) guaranties that the classical inf-sup condition is satisfied. Further, we assume that there exists an integer $r \geq 0$ such that the following approximation properties are satisfied:

$$\| \boldsymbol{\varphi} - \boldsymbol{\Pi}_h \boldsymbol{\varphi} \| \leq C h^i \| \boldsymbol{\varphi} \|_{\boldsymbol{H}^i(\Omega)}, \quad \forall\ \boldsymbol{\varphi} \in \boldsymbol{H}^i(\Omega),\ 1 \leq i \leq r+1, \tag{5}$$

$$\| \psi - Q_h \psi \| \leq C h^i \| \psi \|_{H^i(\Omega)}, \quad \forall\ \psi \in H^i(\Omega),\ 0 \leq i \leq r+1. \tag{6}$$

Furthermore, in our analysis we need the following inverse property:

$$\| \boldsymbol{\varphi}_h \|_{\boldsymbol{V}} \leq C h^{-1} \| \boldsymbol{\varphi}_h \|, \quad \forall\ \boldsymbol{\varphi}_h \in \boldsymbol{V}_h. \tag{7}$$

Here, and throughout the paper, the letter C is used as a generic constant, which is independent of h, p, $\boldsymbol{u}$, etc.

Examples of spaces of piece-wise polynomials that satisfy the conditions stated above are the triangular and rectangular Raviat-Thomas elements from [22] and BDM elements of Brezzi, Douglas and Marini from [4] (for other examples see Brezzi and Fortin [5]). For curved boundary see Douglas and Roberts [10].

Our goal is to prove superconvergence estimates for the mixed finite element approximations using elements of order r. The optimal convergence and superconvergence estimates have been derived by Johnson and Thomée [18] for the case $r = 1$. Recently higher order mixed finite elements with curved boundary have been constructed by Arnold, Douglas and Roberts (see [10]). These elements satisfy the inf-sup condition and hence provide the basis for the analysis in domains with smooth boundary. In this paper, we extend the results of

[18] to the higher order mixed finite element methods. Our analysis is based on the duality argument and uses substantially the estimates in negative norms.

A brief outline of the rest of the paper is as follows. In Section 2, we collect and prove some a priori estimates needed in our analysis. In Section 3, we derive optimal error estimates in the L^2-norm for both p and $\boldsymbol{u}$ provided that the initial data $p(0) \in \dot{H}^i(\Omega)$. The main result in this section, based on several auxiliary lemmas and established in Theorem 1, is

$$\| (p - p_h)(t) \| \leq C h^m t^{-(m-i)/2} \| p(0) \|_{H^i(\Omega)}, \quad 0 \leq i \leq m \leq r+1.$$

Next in Theorem 2 we prove a similar estimate for the velocity $\boldsymbol{u}$:

$$\| (\boldsymbol{u} - \boldsymbol{u}_h)(t) \| \leq C h^m t^{-(m+1-i)/2} \| p(0) \|_{H^i(\Omega)}, \quad 0 \leq i \leq m \leq r+1.$$

The crucial step in the proof of this estimate is establishing a bound for $\| (p - p_h)'(t) \|$ in Lemma 8.

Superconvergence estimates for rectangular finite elements in the L^2-norm are obtained in Section 4 with initial data in $\dot{H}^i(\Omega)$ with $1 \leq i \leq r+2$, where r is the index of the Raviart-Thomas finite element space. Namely, in Theorem 3 and 4 we establish estimates for $|||(Q_h p - p_h)(t)|||$ and $|||(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)(t)|||$, respectively. Here $|||.|||$ denotes a norm that involves the values of the function at the Gaussian points in each finite element. Finally, in Theorem 5 we apply the obtained results and prove superconvergence both for the pressure and the velocity at the Gaussian points for Raviart-Thomas rectangular elements.

2 A priori estimates

Here, we collect some useful a priori estimates for problem (2) and prove some a priori estimates for the mixed finite element equations, which will be used in the following sections.

Lemma 1 *Let $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ be the solution of (2), we have*

$$\| p(t) \|_{H^1(\Omega)}^2 + \int_0^t \| p_s(s) \|^2 \, ds \leq C \| p(0) \|_{H^1(\Omega)}^2 + C \int_0^t \| f(s) \|^2 \, ds; \tag{8}$$

$$\| \boldsymbol{u}(t) \|^2 + \int_0^t \| \boldsymbol{u}(s) \|_{\boldsymbol{H}^1(\Omega)}^2 \, ds \leq C \| p(0) \|_{H^1(\Omega)}^2 + C \int_0^t \| f(s) \|^2 \, ds; \tag{9}$$

$$\| p(t) \|^2 + \int_0^t \left(\| p(s) \|_{H^1(\Omega)}^2 + \| \boldsymbol{u}(s) \|^2 \right) ds \leq C \| p(0) \|^2 + C \int_0^t \| f(s) \|_{H^{-1}(\Omega)}^2 \, ds. \tag{10}$$

If $f = 0$, then, for $p(0) \in \dot{H}^i(\Omega)$, $i \geq 0$,

$$\| \boldsymbol{u}(t) \|_{\boldsymbol{H}^{k-1}(\Omega)} + \| p(t) \|_{H^k(\Omega)} \leq C t^{-(k-i)/2} \| p(0) \|_{H^i(\Omega)}, \quad k \geq i, \tag{11}$$

$$\int_0^t s^{j-i} \left(\| p(s) \|_{H^{j+1}(\Omega)}^2 + \| \boldsymbol{u}(s) \|_{\boldsymbol{H}^j(\Omega)}^2 \right) ds \leq C \| p(0) \|_{H^i(\Omega)}^2, \quad 0 \leq i \leq j. \tag{12}$$

Proof : These estimates are standard for parabolic problems, for instance, see Thomée [25] or Luskin and Rannacher [19]. #

Lemma 2 *Let $p_h \in W_h$ satisfy (3). If $p(0) \in H_0^1(\Omega)$ and*

$$\| (p_h - p)(0) \| \leq Ch \, \| p(0) \|_{H^1(\Omega)},$$

then the following holds:

$$\int_0^t \| p_{h,s}(s) \|^2 \; ds \leq C \, \| p(0) \|^2_{H^1(\Omega)} + C \int_0^t \| f \|^2 \; ds. \tag{13}$$

Proof : Taking $\psi = p_{h,s}$ in the first equation and differentiating the second equation of (2), we have

$$\| p_{h,s} \|^2 + ((\alpha \boldsymbol{u}_h)_s, \boldsymbol{u}_h) = (f, p_{h,s}). \tag{14}$$

Since

$$((\alpha \boldsymbol{u}_h)_s, \boldsymbol{u}_h) = \frac{1}{2}\frac{d}{ds}(\alpha \boldsymbol{u}_h, \boldsymbol{u}_h) + (\alpha_s \boldsymbol{u}_h, \boldsymbol{u}_h),$$

and integrating (14), we conclude that

$$\| \boldsymbol{u}_h(t) \|^2 + \int_0^t \| p_{h,s} \|^2 \; ds \leq C \, \| \boldsymbol{u}_h(0) \|^2 + C \int_0^t \left(\| \boldsymbol{u}_h \|^2 + \| f \|^2 \right) \; ds. \tag{15}$$

In order to estimate $\int_0^t \| \boldsymbol{u}_h \|^2 \, ds$, we take $\psi_h = p_h$, $\boldsymbol{\varphi}_h = \boldsymbol{u}_h$ in (3) to get

$$(\alpha \boldsymbol{u}_h, \boldsymbol{u}_h) + (p_{h,s}, p_h) = (f, p_h).$$

Hence, we have

$$\alpha_0 \, \| \boldsymbol{u}_h \|^2 + \frac{1}{2}\frac{d}{ds} \, \| p_h \|^2 \leq C_\varepsilon \, \| f \|^2 + \varepsilon \, \| p_h \|^2, \tag{16}$$

where $\varepsilon > 0$ is a fixed number. It can be easily verified that $\| p_h \| \leq C \, \| \boldsymbol{u}_h \|$. Thus, by (16) with sufficiently small ε, we have

$$\int_0^t \| \boldsymbol{u}_h \|^2 \, ds + \| p_h(t) \|^2 \leq C \int_0^t \| f \|^2 \, ds + C \, \| p(0) \|^2 \, . \tag{17}$$

By virtue of

$$\begin{aligned} (\alpha \boldsymbol{u}_h(0), \boldsymbol{u}_h(0)) &= -(\nabla \cdot \boldsymbol{u}_h(0), p_h(0)) \\ &= -(\nabla \cdot \boldsymbol{u}_h(0), p(0)) + (\nabla \cdot \boldsymbol{u}_h(0), p(0) - p_h(0)), \end{aligned}$$

and the inverse property (7) we conclude that

$$\| \boldsymbol{u}_h(0) \| \leq C \, \| \nabla p(0) \|,$$

which with (15) and (17) completes the proof. #

3 L^2-error estimates

In this section we provide all preliminaries necessary for deriving the superconvergence results obtained in the next section. The main results are contained in Theorem 1 and Theorem 2, where L^2- estimates for $p-p_h$ and $\boldsymbol{u}-\boldsymbol{u}_h$ are derived for homogeneous equations with initial data from $\dot{H}^i(\Omega)$, $0 \le i \le r+1$.

We first introduce some notations. Let $E(t_1, t_0)$, $t_0 \le t_1$ denote the solution operator of the homogeneous parabolic equation, i.e., for $\psi \in L^2(\Omega)$, $q(t_1) = E(t_1, t_0)\psi \in H_0^1(\Omega)$ is the weak solution of the initial value problem:

$$q_t - \nabla \cdot (a \nabla q) = 0, \quad t_0 < t, \quad q(t_0) = \psi.$$

Similarly, $E^*(t_1, t_0)$, $t_0 \le t_1$ denotes the solution operator of the corresponding backward running in time problem: i.e., for $\psi \in L^2(\Omega)$, $q(t_0) = E^*(t_1, t_0)\psi \in H_0^1(\Omega)$ is the weak solution of

$$q_t + \nabla \cdot (a \nabla q) = 0, \quad t < t_1, \quad q(t_1) = \psi.$$

Let $(q_h, \boldsymbol{v}_h) \in W_h \times \boldsymbol{V}_h$ denote the mixed finite element approximation of the pair $(q, a\nabla q)$ with initial approximation ψ_h:

$$\begin{cases} (q_{h,t}, \psi_h) - (\nabla \cdot \boldsymbol{v}_h, \psi_h) = 0, & \forall \psi_h \in W_h, \ t_0 < t, \\ (\alpha \boldsymbol{v}_h, \boldsymbol{\varphi}_h) + (\nabla \cdot \boldsymbol{\varphi}_h, q_h) = 0, & \forall \ \boldsymbol{\varphi}_h \in \boldsymbol{V}_h, t_0 < t, \\ q_h(t_0) = \psi_h. \end{cases}$$

Define now $E_h(t_1, t_0)\psi_h = q_h(t_1)$ as the mixed finite element analogy of $E(t_1, t_0)\psi$. In a similar way, we define $E_h^*(t_1, t_0)\psi_h$ as the discrete analogy of $E^*(t_1, t_0)\psi$. Further, we introduce the error operators $F_h(t_1, t_0) = E(t_1, t_0) - E_h(t_1, t_0)Q_h$ and $F_h^*(t_1, t_0) = E^*(t_1, t_0) - E_h^*(t_1, t_0)Q_h$. One can easily verify that the following identity holds true:

$$(F_h(t_1, t_0)\psi_1, \psi_2) = (\psi_1, F_h^*(t_1, t_0)\psi_2). \tag{18}$$

3.1 L^2 estimate for $(p - p_h)(t)$

In this section we derive an L^2-estimate for $p - p_h$ in the case when $p(0) \in \dot{H}^i(\Omega)$, where $0 \le i \le r+1$. Note that the case $i = 0$ corresponds to initial data $p(0)$ just in $L^2(\Omega)$. This estimate will be established in several steps.

First we establish the preliminary estimate of Lemma 3, and then in Lemma 4, we estimate $p - p_h$ in $H^{-j}(\Omega)$, $0 \le j \le r+1$.

Lemma 3 *Let* $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ *and* $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ *be the solutions of problems* (2) *and* (3)*, respectively. If* $p_h(0) = Q_h p(0)$*, then*

$$\| (Q_h p - p_h)(t) \|^2 + \int_0^t \| \boldsymbol{u} - \boldsymbol{u}_h \|^2 \, ds \leq C \int_0^t \| \boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u} \|^2 \, ds \tag{19}$$

Proof : From (2) and (3), we have the following error equations:

$$(p_s - p_{h,s}, \psi_h) - (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h), \psi_h) = 0, \quad \forall \, \psi_h \in W_h, \tag{20}$$

$$(\nabla \cdot \boldsymbol{\varphi}_h, p - p_h) + (\alpha(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{\varphi}_h) = 0, \quad \forall \, \boldsymbol{\varphi}_h \in \boldsymbol{V}_h. \tag{21}$$

Taking $\psi_h = Q_h p - p_h$ and $\boldsymbol{\varphi}_h = \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h$ in (20) and (21), we obtain

$$\frac{1}{2} \frac{d}{ds} \| Q_h p - p_h \|^2 + \alpha_0 \| \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h \|^2 \leq (\alpha(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h). \tag{22}$$

Applying Schwarz inequality to the right hand side of (22), we get

$$\frac{d}{ds} \| Q_h p - p_h \|^2 + \alpha_0 \| \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h \|^2 \leq C \| \boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u} \|^2 . \tag{23}$$

Integrating this over $[0, t]$, and noting that $p_h(0) = Q_h p(0)$, we obtain the desired estimate (19). #

In the next lemmas, we establish error estimates for $p - p_h$ in negative norms which play essential role in our analysis.

Lemma 4 *Let* $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ *and* $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ *be the solutions of problems* (2) *and* (3)*, respectively. If* $f = 0, p(0) \in \dot{H}^i(\Omega)$ *and* $p_h(0) = Q_h p(0)$*, then*

$$\| (p - p_h)(t) \|_{\dot{H}_t^{-j}(\Omega)} \leq C h^{i+j} \| p(0) \|_{H^i(\Omega)}, \quad 0 \leq i, j \leq r+1. \tag{24}$$

Proof : Applying Lemma 3 and the approximation properties (5) and (6) for $0 \leq i \leq r+1$, we get

$$\| (p - p_h)(t) \| = \| F_h(t, 0) p(0) \| \leq C h^i \left(\| p(t) \|_{H^i(\Omega)} + \int_0^t \| \boldsymbol{u} \|^2_{\boldsymbol{H}^i(\Omega)} \, ds \right)^{1/2} \leq C h^i \| p(0) \|_{H^i(\Omega)},$$

which proves (24) with $j = 0$. This estimate can be applied also to the solution of the backward in time adjoint parabolic problem with initial data $\psi \in \dot{H}_t^j(\Omega)$ that is expressed in term of the operators F_h^*:

$$\| F_h^*(t, s) \psi \| \leq C h^j \| \psi \|_{H^j(\Omega)}, \quad 0 \leq j \leq r+1, \; s < t.$$

Using this estimate and identity (18) we get

$$\begin{aligned} ((p - p_h)(t), \psi) &= (F_h(t, 0) p(0), \psi) = (p(0), F_h^*(t, 0) \psi) \\ &\leq \| p(0) \| \| F_h^*(t, 0) \psi \| \leq C h^j \| p(0) \| \| \psi \|_{H^j(\Omega)}, \end{aligned} \tag{25}$$

which yields (24) with $i=0$. It remains to prove (24) for $1 \le i,j \le r+1$. For any $\psi \in \dot{H}_t^j(\Omega)$, let $q(s) = E^*(t,s)\psi$, and $\boldsymbol{v} = a\nabla q$. For $s < t$, let $(q_h(s), \boldsymbol{v}_h(s)) \in W_h \times \boldsymbol{V}_h$ be the mixed finite element approximation of the pair $(q, \boldsymbol{v})$ with $q_h(t) = Q_h\psi$. Then, we have

$$
\begin{aligned}
((Q_hp - p_h)(t), \psi) &= \int_0^t \frac{d}{ds}(Q_hp - p_h, q_h)ds = \int_0^t ((p_s - p_{h,s}, q_h) + (Q_hp - p_h, q_{h,s}))\, ds \\
&= \int_0^t ((\nabla \cdot (\boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h), q_h) - (\nabla \cdot \boldsymbol{v}_h, p - p_h))\, ds \\
&= \int_0^t (-(\alpha\boldsymbol{v}_h, \boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h) + (\alpha(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{v}_h))\, ds \\
&= \int_0^t (\alpha\boldsymbol{v}_h, \boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})ds \\
&= \int_0^t ((\alpha\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}) + (\alpha(\boldsymbol{v}_h - \boldsymbol{v}), \boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}))\, ds \\
&= \int_0^t \{(p_s - Q_hp_s, q - Q_hq) + (\alpha(\boldsymbol{v}_h - \boldsymbol{v}), \boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})\}\, ds. \qquad (26)
\end{aligned}
$$

Here, in the last step, we have used the following fact:

$$
\begin{aligned}
(\alpha\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}) &= -(q - Q_hq, \nabla \cdot (\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})) = -(\nabla \cdot \boldsymbol{u}, q - Q_hq) \\
&= (p_s, q - Q_hq) = (p_s - Q_hp_s, q - Q_hq).
\end{aligned}
$$

By Lemma 3, one gets

$$
\begin{aligned}
\int_0^t (\alpha(\boldsymbol{v}_h - \boldsymbol{v}), \boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})\, ds &\le Ch^{i+j} \left(\int_0^t \|\boldsymbol{v}\|^2_{\boldsymbol{H}^j(\Omega)}\, ds\right)^{1/2} \left(\int_0^t \|\boldsymbol{u}\|^2_{\boldsymbol{H}^i(\Omega)}\, ds\right)^{1/2} \\
&\le Ch^{i+j}\, \|\psi\|_{H^j(\Omega)} \|p(0)\|_{H^i(\Omega)},
\end{aligned}
$$

where $1 \le i,j \le r+1$. Further, in order to use the smoothing properties of the parabolic operator, we estimate the first term on the right hand side of equation (26) by splitting the interval $(0,t)$ into two subintervals, e.g., $(0,t/2)$ and $(t/2,t)$. The solution p is smooth in $(t/2,t)$, while the solution q is smooth in $(0,t/2)$. Therefore, using (6), the inequality $\|p_s(t)\|_{H^i(\Omega)} \le C\, \|p(t)\|_{H^{i+2}(\Omega)}$ and (11), we get

$$
\begin{aligned}
\int_{t/2}^t (p_s - Q_hp_s, q - Q_hq)ds &\le Ch^{i+j} \int_{t/2}^t \|p_s\|_{H^i(\Omega)} \|q\|_{H^j(\Omega)}\, ds \\
&\le Ch^{i+j}\, \|\psi\|_{H^j(\Omega)} \|p(0)\|_{H^i(\Omega)},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{t/2} (p_s - Q_hp_s, q - Q_hq)ds &= (p(t/2) - Q_hp(t/2), q(t/2) - Q_hq(t/2)) \\
&\quad -(p(0) - Q_hp(0), q(0) - Q_hq(0)) \\
&\quad - \int_0^{t/2} (p - Q_hp, q_s - Q_hq_s)ds \\
&\le Ch^{i+j}\, \|\psi\|_{H^j(\Omega)} \|p(0)\|_{H^i(\Omega)}\, .
\end{aligned}
$$

Hence, it is shown that

$$((p-p_h)(t), \psi) \le Ch^{i+j} \, \|\psi\|_{H^j(\Omega)} \|p(0)\|_{H^i(\Omega)},$$

which proves the desired assertion. #

Obviously, the established estimate (24) is not useful for $i = j = 0$, i.e., for initial data $p(0) \in L^2(\Omega)$. However, this estimate plays an essential role in the proof of the main result established in theorem 1. We begin with the follow lemma:

Lemma 5 *Let* $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ *and* $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ *be the solutions of problems* (2) *and* (3)*, respectively. If* $f = 0$ *and* $p_h(0) = Q_h p(0)$*,* $p(0) \in L^2(\Omega)$*, then*

$$\int_0^t \|p - p_h\|^2 \, ds \le Ch^2 \, \|p(0)\|^2 \,. \tag{27}$$

Proof : Let $q(s) \in H_0^1(\Omega)$ be the solution of the following backward in time parabolic problem:

$$\begin{cases} q_s + \nabla \cdot (a \nabla q) = Q_h p - p_h, \quad 0 \le s < t, \\ q(t) = 0. \end{cases} \tag{28}$$

Since the right hand side is in $L^2(\Omega)$ and the initial data and boundary condition are compatible, we have $q(s) \in H^2(\Omega)$. Set $\boldsymbol{v} = a\nabla q$. Let $(q_h, \boldsymbol{v}_h) \in W_h \times \boldsymbol{V}_h$ be the mixed finite element approximation to $(q, \boldsymbol{v})$ with $q_h(t) = 0$, i.e.

$$\begin{cases} (q_{h,s}, \psi_h) + (\nabla \cdot \boldsymbol{v}_h, \psi_h) = (Q_h p - p_h, \psi_h), & \forall\, \psi_h \in W_h, \ s < t, \\ (\alpha \boldsymbol{v}_h, \boldsymbol{\varphi}_h) + (q_h, \nabla \cdot \boldsymbol{\varphi}_h) = 0, & \forall\, \boldsymbol{\varphi}_h \in V_h, \ s < t. \end{cases} \tag{29}$$

According to the above definition, it follows that

$$\begin{aligned} \|Q_h p - p_h\|^2 &= (q_{h,s}, Q_h p - p_h) + (\nabla \cdot \boldsymbol{v}_h, Q_h p - p_h) \\ &= \frac{d}{ds}(q_h, Q_h p - p_h) - (q_h, p_s - p_{h,s}) + (\nabla \cdot \boldsymbol{v}_h, p - p_h) \\ &= \frac{d}{ds}(q_h, Q_h p - p_h) - (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h), q_h) - (\alpha(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{v}_h) \\ &= \frac{d}{ds}(q_h, Q_h p - p_h) + (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h). \end{aligned}$$

Integrating the above equality over the interval $[0, t]$ and noting that $q_h(t) = 0$ and $(Q_h p - p_h)(0) = 0$, we have

$$\begin{aligned} \int_0^t \|Q_h p - p_h\|^2 \, ds &= \int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h) \, ds \\ &= \int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}) \, ds + \int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h - \boldsymbol{v}) \, ds. \end{aligned} \tag{30}$$

Below, we estimate the two terms on the right hand side of (30) separately. We first consider the term $\int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v})\, ds$. By virtue of $\boldsymbol{v} = a\nabla q$, we have

$$\begin{aligned}
\int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v})\, ds &= -\int_0^t (\nabla \cdot (\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), q)\, ds \\
&= \int_0^t (\nabla \cdot \boldsymbol{u} - Q_h \nabla \cdot \boldsymbol{u}, q)\, ds = \int_0^t (p_s, q - Q_h q)\, ds \\
&= -(p(0), q(0) - Q_h q(0)) - \int_0^t (p - Q_h p, q_s)\, ds \\
&\leq Ch \, \| p(0) \| \cdot \| q(0) \|_{H^1(\Omega)} + Ch \int_0^t \| p \|_{H^1(\Omega)} \| q_s \| \; ds.
\end{aligned}$$

Since q is the solution of (28), i.e., a solution of a backward in time parabolic problem with homogeneous initial condition and right hand side $Q_h p - p_h$, then the corresponding a priori estimate (8) leads us to

$$\| q(0) \|_{H^1(\Omega)}^2 + \int_0^t \| q_s(s) \|^2 \; ds \leq C \int_0^t \| Q_h p - p_h \|^2 \; ds.$$

This, then, implies that

$$\int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v})\, ds \leq Ch \, \| p(0) \| \left(\int_0^t \| Q_h p - p_h \|^2 \; ds \right)^{1/2}. \tag{31}$$

Next, we estimate the second term $\int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h - \boldsymbol{v})\, ds$ involved on the right hand side of (30). To this end, we divide the integration over two subintervals:

$$\begin{aligned}
\int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h - \boldsymbol{v})\, ds &= \int_{h^2}^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h - \boldsymbol{v})\, ds \\
&\quad + \int_0^{h^2} (\alpha \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{v}_h)\, ds + \int_0^{h^2} (\alpha \mathbf{\Pi}_h \boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{v})\, ds.
\end{aligned}$$

Applying Lemma 3 for $\boldsymbol{v} - \boldsymbol{v}_h$ and estimates (5), we infer that

$$\begin{aligned}
\int_{h^2}^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h - \boldsymbol{v})\, ds &\leq C \left(\int_{h^2}^t \| \boldsymbol{u} - \mathbf{\Pi}_h \boldsymbol{u} \|^2 \; ds \right)^{1/2} \left(\int_{h^2}^t \| \boldsymbol{v} - \boldsymbol{v}_h \|^2 \; ds \right)^{1/2} \\
&\leq Ch^2 \left(\int_{h^2}^t \| \boldsymbol{u} \|_{\boldsymbol{H}^1(\Omega)}^2 \; ds \right)^{1/2} \left(\int_{h^2}^t \| \boldsymbol{v} \|_{\boldsymbol{H}^1(\Omega)}^2 \; ds \right)^{1/2}.
\end{aligned}$$

Then, the a priori estimate (9) applied to the solution $\boldsymbol{v}$ of (28), which is the backward in time nonhomogeneous parabolic equation with homogeneous initial data, and the estimate (11) for $\boldsymbol{u}$, lead to

$$\begin{aligned}
\int_{h^2}^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h - \boldsymbol{v})\, ds &\leq Ch^2 \, \| p(0) \| \left(\int_{h^2}^t s^{-2}\, ds \right)^{1/2} \left(\int_{h^2}^t \| Q_h p - p_h \|^2 \; ds \right)^{1/2} \\
&\leq Ch \, \| p(0) \| \left(\int_{h^2}^t \| Q_h p - p_h \|^2 \; ds \right)^{1/2}.
\end{aligned}$$

Using the same argument as above, we have

$$\begin{aligned}\int_0^{h^2} (\alpha \boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{v})\, ds &\leq C \left(\int_0^{h^2} \|\boldsymbol{u}\|^2 \, ds \right)^{1/2} \left(\int_0^{h^2} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 \, ds \right)^{1/2} \\ &\leq Ch \|p(0)\| \left(\int_0^t \|Q_h p - p_h\|^2 \, ds \right)^{1/2}.\end{aligned}$$

Now we estimate the last term. Unfortunately, the projection operator $\mathbf{\Pi}_h$ is not stable in L^2. Thus, we have to use the fact the pair $(q, \boldsymbol{v})$ is the solution of problem (28) and $(q_h, \boldsymbol{v}_h)$ is its mixed finite element approximation defined by (29): In order to estimate the last term, we first note that , by definition,

$$\begin{aligned}\int_0^{h^2} (\alpha \mathbf{\Pi}_h \boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{v})\, ds &= \int_0^{h^2} (\nabla \cdot \mathbf{\Pi}_h \boldsymbol{u}, q - q_h)\, ds \\ &= -\int_0^{h^2} (\nabla \cdot \boldsymbol{u}, q_h - Q_h q)\, ds.\end{aligned}$$

Then, integrating by parts and using the a priopri estimates (8) and (9) for $\boldsymbol{v}$, as well as (13) in Lemma 2 for $\boldsymbol{v}_h$, we conclude that

$$\begin{aligned}&\int_0^{h^2} (\alpha \mathbf{\Pi}_h \boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{v})\, ds = \int_0^{h^2} (p_s, Q_h q - q_h)\, ds \\ &= (p(h^2), Q_h q(h^2) - q_h(h^2)) - (p(0), Q_h q(0) - q_h(0)) + \int_0^{h^2} (p, q_{h,s} - Q_h q_s)\, ds \\ &\leq Ch \|p(0)\| \left(\int_0^t \|\boldsymbol{v}(s)\|^2_{\boldsymbol{H}^1(\Omega)} \, ds \right)^{1/2} + Ch \|p(0)\| \left(\int_0^{h^2} (\|q_{h,s}\|^2 + \|q_s\|^2)\, ds \right)^{1/2} \\ &\leq Ch \|p(0)\| \left(\int_0^t \|Q_h p - p_h\|^2 \, ds \right)^{1/2}.\end{aligned}$$

Combining the above three estimates with (32), we conclude that

$$\int_0^t (\alpha(\mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h - \boldsymbol{v})\, ds \leq Ch \|p(0)\| \left(\int_0^t \|Q_h p - p_h\|^2 \, ds \right)^{1/2}. \tag{32}$$

Thus, substituting the estimates (31) and (32) into (30), we obtain

$$\int_0^t \|Q_h p - p_h\|^2 \, ds \leq Ch^2 \|p(0)\|^2 . \tag{33}$$

Finally, in virtue of (6) and (12), we have

$$\int_0^t \|Q_h p - p\|^2 \, ds \leq Ch^2 \|p(0)\|^2,$$

which, together with (33), leads to the desired estimate. #

Lemma 6 *Under the conditions of Lemma 5,*

$$\| (p - p_h)(t) \| \leq Cht^{-1/2} \, \| p(0) \| \, . \tag{34}$$

Proof : We start with multiplying by s the inequality (23):

$$\frac{d}{ds}\left(s \, \| Q_h p - p_h \|^2\right) + s \, \| \mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h \|^2 \leq Cs \, \| \boldsymbol{u} - \mathbf{\Pi}_h \boldsymbol{u} \|^2 + \| Q_h p - p_h \|^2 \, .$$

Next, integrating over $[0, t]$, we get

$$\begin{aligned} t \, \| (Q_h p - p_h)(t) \|^2 + \int_0^t s \, \| \mathbf{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h \|^2 \, ds \quad &\leq \quad C \int_0^t s \, \| \boldsymbol{u} - \mathbf{\Pi}_h \boldsymbol{u} \|^2 \, ds \\ &\quad + C \int_0^t \| Q_h p - p_h \|^2 \, ds. \end{aligned}$$

Consequently, applying (6), (12) and Lemma 5 to the above estimates proves the desired estimate (34). #

Now, we have all preliminaries needed to prove the main result of this section stated in the following theorem.

Theorem 1 *Let* $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ *and* $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ *be the solutions of problems* (2) *and* (3)*, respectively. If* $f = 0, p(0) \in \dot{H}^i(\Omega)$ *and* $p_h(0) = Q_h p(0)$*, then*

$$\| (p - p_h)(t) \| \leq Ch^m t^{-(m-i)/2} \, \| p(0) \|_{H^i(\Omega)}, \tag{35}$$

where $0 \leq i \leq m \leq r + 1$.

Proof : First, we shall prove (35) for $i = 0$ and $0 < m \leq r + 1$. Obviously, for $m = 1$ the inequality (35) is simply the result of Lemma 6. Then we prove (35) consequently for $m = 2, ..., r + 1$ (for $i = 0$). To this end, we note that the solutions $p(s), p_h(s)$ of the forward parabolic problem and the solutions $q(s) = E^*(t, s)\psi, q_h = E_h^*(t, s)Q_h\psi$ of the corresponding backward in time adjoint problem satisfy the identities:

$$\frac{d}{ds}\,(p(s), q(s)) = 0, \qquad \frac{d}{ds}\,(p_h(s), q_h(s)) = 0.$$

For $\psi \in L^2(\Omega)$, we subtract these identities and integrate for $s \in (t/2, t)$ to get

$$\begin{aligned} ((p - p_h)(t), \psi) \quad &= \quad (p(t/2), E^*(t, t/2)\psi) - (p_h(t/2), E_h^*(t, t/2)Q_h\psi) \\ &= \quad (F_h(t/2, 0)p(0), E^*(t, t/2)\psi) + (p(t/2), F_h^*(t, t/2)\psi) \\ &\quad\quad - (F_h(t/2, 0)p(0), F_h^*(t, t/2)\psi). \end{aligned}$$

Next, applying the identity (18), we obtain

$$\begin{aligned} ((p - p_h)(t), \psi) \quad &= \quad (p(0), F_h^*(t/2, 0)E^*(t, t/2)\psi) + (\psi, F_h(t, t/2)p(t/2)) \\ &\quad\quad + (F_h(t/2, 0)p(0), F_h^*(t, t/2)\psi). \end{aligned} \tag{36}$$

In view of Lemma 4, (36) results in

$$\begin{aligned}((p-p_h)(t),\psi) \ \leq\ & Ch^i \,\|p(0)\|\|E^*(t,t/2)\psi\|_{H^i(\Omega)} + Ch^i \,\|\psi\|\|p(t/2)\|_{H^i(\Omega)} \\ & + \|(p-p_h)(t/2)\|\|F_h^*(t,t/2)\psi\|, \end{aligned} \tag{37}$$

where $0 \leq i \leq r+1$. Thus, taking into account of Lemma 6 and the a priori estimate (11), we conclude, from (37), that

$$\|(p-p_h)(t)\| \leq Ch^2 t^{-1} \,\|p(0)\|, \tag{38}$$

which is the required result for $i=0$ and $m=2$. Further, if $r>1$, applying the estimate (37) and (38) as well as Lemma 4, we get

$$\|(p-p_h)(t)\| \leq Ch^3 t^{-3/2} \,\|p(0)\|\,. \tag{39}$$

Repeating this procedure, we prove the estimate (35) for any $0 < m \leq r+1$ with $i=0$. To show (35) for $i>0$, we examine (36) again. Applying Lemma 4 and the estimate (35) proved above for $i=0$, we have

$$\begin{aligned}((p-p_h)(t),\psi) \ \leq\ & \|p(0)\|_{H^i(\Omega)}\|F_h^*(t/2,0)E^*(t,t/2)\psi\|_{\dot{H}_t^{-i}(\Omega)} \\ & + \|\psi\|\|F_h(t,t/2)p(t/2)\| + \|p_h(t/2)-p(t/2)\|\|F^*(t,t/2)\psi\| \\ \leq\ & Ch^m \,\|p(0)\|_{H^i(\Omega)}\|E^*(t,t/2)E^*(t/2,0)\psi\|_{H^{m-i}(\Omega)} \\ & + Ch^m \,\|\psi\|\|p(t)\|_{H^m(\Omega)} + Ch^i \,\|p(0)\|_{H^i(\Omega)} \cdot Ch^{m-i}t^{-(m-i)/2} \,\|\psi\| \\ \leq\ & Ch^m t^{-(m-i)/2} \,\|\psi\|\,. \end{aligned}$$

This completes the proof. #

3.2 L^2 estimates for $u - u_h$

This subsection is a continuation of the previous subsection. Below, for a function $\theta(t)$, we denote its kth derivative with respect to t as $\theta^{(k)}(t)$.

Lemma 7 *Let $(p,\boldsymbol{u}) \in W \times \boldsymbol{V}$ and $(p_h,\boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ be the solutions of problems* (2) *and* (3)*, respectively. If $p_h(0) = Q_h p(0)$, $p(0) \in \dot{H}^i(\Omega)$, then,*

$$t^\gamma \,\|(Q_h p - p_h)(t)\|^2 + \int_0^t s^\gamma \,\|\boldsymbol{u}-\boldsymbol{u}_h\|^2 \ ds \leq Ch^m t^{\gamma-m+i} \,\|p(0)\|_{H^i(\Omega)}, \tag{40}$$

and

$$\begin{aligned} & t^\gamma \,\|(Q_h p - p_h)'(t)\|^2 + \int_0^t s^\gamma \,\|(\boldsymbol{u}-\boldsymbol{u}_h)'\|^2 \ ds \\ \leq\ & Ch^m t^{\gamma-m+i-2} \,\|p(0)\|^2_{H^i(\Omega)} + C\int_0^t s^{\gamma-1} \,\|(p-p_h)'\|^2 \ ds, \end{aligned} \tag{41}$$

where $0 \leq i \leq m \leq r+1$, $\gamma > m-i+2$.

Proof : The estimate (40) is a simple consequence of (23), (35), and (11). We multiply (40) by s and intergrate over $[0,t]$ to get

$$\begin{aligned} & t^{\gamma} \parallel (Q_h p - p_h)(t) \parallel^2 + \int_0^t s^{\gamma} \parallel \boldsymbol{u} - \boldsymbol{u}_h \parallel^2 \; ds \\ \leq & \; C \int_0^t s^{\gamma} \parallel \boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u} \parallel^2 ds + C \int_0^t s^{\gamma-1} \parallel p - Q_h p \parallel^2 ds + C \int_0^t s^{\gamma-1} \parallel p - p_h \parallel^2 ds \\ \leq & \; C h^m t^{\gamma - m + i} \parallel p(0) \parallel_{H^i(\Omega)} . \end{aligned}$$

To prove (41) we differentiate (20) and (21) and take

$$\psi_h = (Q_h p - p_h)', \qquad \boldsymbol{\varphi}_h = (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)'$$

to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{ds} \parallel (Q_h p - p_h)' \parallel^2 + \alpha_0 \parallel (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)' \parallel^2 \\ \leq & \; ((\alpha(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}))', (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)') + (\alpha'(\boldsymbol{u}_h - \boldsymbol{\Pi}_h \boldsymbol{u}), (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)') , \end{aligned}$$

which obviously yields

$$\begin{aligned} & \frac{d}{ds} \parallel (Q_h p - p_h)' \parallel^2 + \alpha_0 \parallel (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)' \parallel^2 \\ \leq & \; C \sum_{j=0}^{1} \parallel (\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u})^{(j)} \parallel^2 + C \parallel \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h \parallel^2 . \end{aligned} \tag{42}$$

By multiplying (42) by s^{γ}, and then, by integrating over the interval $[0,t]$, we get

$$\begin{aligned} & t^{\gamma} \parallel (Q_h p - p_h)'(t) \parallel^2 + \alpha_0 \int_0^t s^{\gamma} \parallel (\boldsymbol{u} - \boldsymbol{u}_h)' \parallel^2 \; ds \\ \leq & \; C \sum_{j=0}^{1} \int_0^t s^{\gamma} \parallel (\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u})^{(j)} \parallel^2 ds + C \int_0^t s^{\gamma-1} \parallel (p - Q_h p)' \parallel^2 ds \\ & + C \int_0^t s^{\gamma-1} \parallel (p - p_h)' \parallel^2 ds + C \int_0^t s^{\gamma} \parallel \boldsymbol{u} - \boldsymbol{u}_h \parallel^2 ds \\ \leq & \; C h^{2m} t^{\gamma - m + i - 2} \parallel p(0) \parallel^2_{H^i(\Omega)} + C \int_0^t s^{\gamma-1} \parallel (p - p_h)' \parallel^2 \; ds \\ & + C \int_0^t s^{\gamma} \parallel \boldsymbol{u} - \boldsymbol{u}_h \parallel^2 ds. \end{aligned}$$

Next, we estimate $\int_0^t s^{\gamma} \parallel \boldsymbol{u} - \boldsymbol{u}_h \parallel^2 ds$ by (40) and get the required inequality (41). #

Before proving the main result of this section we first establish an estimate for $(p-p_h)'(t)$.

Lemma 8 *If $f=0$, $p_h(0) = Q_h p(0)$, and $p(0) \in \dot{H}^i(\Omega)$, then,*

$$\parallel (p - p_h)'(t) \parallel \leq C h^m t^{-(m+2-i)/2} \parallel p(0) \parallel_{H^i(\Omega)} . \tag{43}$$

Proof : For $\psi \in L^2(\Omega)$, let $q(s) = E^*(t,s)\psi$ and $\boldsymbol{v}(s) = a\nabla q$. Also Let $(q_h, \boldsymbol{v}_h) \in W_h \times \boldsymbol{V}_h$ be the mixed finite element approximation of pair $(q, \boldsymbol{v})$. Our starting point in the proof will be the following fundamental identity, which is used also in the superconvergence analysis considered in the next section: for $s < t$

$$\begin{aligned} ((Q_h p - p_h)'(t), \psi) &= ((Q_h p - p_h)(s), q_h(s))' - ((Q_h p - p_h)(s), q_{h,s}(s)) \\ &+ \int_s^t \Big((\alpha(\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}))', \boldsymbol{v}_h \Big)\, d\xi + \int_s^t (\alpha'(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{v}_h)\, d\xi. \end{aligned} \tag{44}$$

To show that we use the fact that p, p_h, q, and q_h are solutions to certain parabolic problems. Since $q_h(t) = Q_h\psi$ we first write the following obvious equality for $s < t$:

$$\begin{aligned} ((Q_h p - p_h)'(t), \psi) &= ((Q_h p - p_h)(s), q_h(s))' - ((Q_h p - p_h)(s), q_{h,s}(s)) \\ &+ \int_s^t ((Q_h p - p_h)'(\xi), q_h(\xi))'\, d\xi. \end{aligned} \tag{45}$$

Now we transform the expression under the integral sign

$$((Q_h p - p_h)'(\xi), q_h(\xi))' = (Q_h p''(\xi) - p_h''(\xi), q_h(\xi)) + (Q_h p'(\xi) - p_h'(\xi), q_h'(\xi)).$$

Taking into account that p and p_h are solution to the problems (2) and (3), respectively (with $f = 0$), we transform the first term in the following way

$$\begin{aligned} ((Q_h p - p_h)''(\xi), q_h(\xi)) &= (p''(\xi) - p_h''(\xi), q_h(\xi)) = (\nabla.(\boldsymbol{u}'(\xi) - \boldsymbol{u}_h'(\xi)), q_h(\xi)) \\ &= (\nabla.(\boldsymbol{\Pi}_h \boldsymbol{u}'(\xi) - \boldsymbol{u}_h'(\xi)), q_h(\xi)) \\ &= (\alpha(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h))'(\xi), \boldsymbol{v}_h(\xi)). \end{aligned}$$

The second term is transformed in a similar manner:

$$((Q_h p - p_h)'(\xi), q_h'(\xi)) = -(Q_h p'(\xi) - p_h'(\xi), \nabla.\boldsymbol{v}_h(\xi)) = (((\alpha(\boldsymbol{u} - \boldsymbol{u}_h))'(\xi), \boldsymbol{v}_h(\xi)).$$

This completes the proof of (44).

The terms involved in the right hand side of (44) are estimated below. Applying (5) and (11), and using Lemma 7, we have, for $s \in [t/2, t]$, that

$$\begin{aligned} &\left| \int_s^t \Big(\alpha' (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{v}_h \Big)\, d\xi \right| \\ \leq\; & Ct^{-\gamma/2} \left(\int_{t/2}^t \xi^\gamma \, \| \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u} \|^2 \, d\xi \right)^{1/2} \left(\int_{t/2}^t \| \boldsymbol{v}_h \|^2 \, d\xi \right)^{1/2} \\ & + Ct^{-\gamma/2} \left(\int_{t/2}^t \xi^\gamma \, \| \boldsymbol{u} - \boldsymbol{u}_h \|^2 \, d\xi \right)^{1/2} \left(\int_{t/2}^t \| \boldsymbol{v}_h \|^2 \, d\xi \right)^{1/2} \\ \leq\; & Ct^{-\gamma/2} \, \| \psi \| \left(Ch^{2m} t^{\gamma - m + i} \, \| p(0) \|_{H^i(\Omega)}^2 + \int_0^t \xi^{\gamma - 1} \, \| p - p_h \|^2 \, d\xi \right)^{1/2} \\ \leq\; & Ch^m t^{-(m-i)/2} \, \| p(0) \|_{H^i(\Omega)} \| \psi \|, \end{aligned} \tag{46}$$

where γ is taken appropriately large. Then, using the estimate (35) of Theorem 1, we get

$$|\,((Q_hp-p_h)(s),q_{h,s}(s))\,|\leq Ch^m s^{-(m-i)/2}(t-s)^{-1}\,\|\,p(0)\,\|_{H^i(\Omega)}\|\,\psi\,\|\,. \tag{47}$$

Furthermore, using (5), (6) and (19) of Lemma 3, we assert that

$$\begin{aligned}\int_s^t((\alpha(\boldsymbol{u}-\mathbf{\Pi}_h\boldsymbol{u}))',\boldsymbol{v}_h)\,d\xi &\leq Ch^m t^{-(m-i)/2-1}\,\|\,p(0)\,\|_{H^i(\Omega)}\left(\int_0^t\|\,\boldsymbol{v}_h\,\|^2\,d\xi\right)^{1/2}\\ &\leq Ch^m t^{-(m-i)/2-1}\,\|\,p(0)\,\|_{H^i(\Omega)}\|\,\psi\,\|\,. \end{aligned}\tag{48}$$

Multiplying (44) by $t-s$, and using (46), (47) and (48), we obtain, for $s\in[t/2,t]$, that

$$\begin{aligned}(t-s)\,((Q_hp-p_h)'(t),\psi) &\leq [(t-s)\,((Q_hp-p_h)(s),q_h(s))]'\,[(t-s)\,((Q_hp-p_h)(s),q_h(s))]'\\ &\quad +Ch^m t^{-(m-i)/2}\,\|\,p(0)\,\|_{H^i(\Omega)}\|\,\psi\,\|\,.\end{aligned}$$

Integrating this inequality for s over the interval $[t/2,t]$, leads us to

$$\begin{aligned}\frac{1}{2}\left(\frac{t}{2}\right)^2\left((Q_hp-p_h)'(t),\psi\right) &\leq -\frac{t}{2}\,((Q_hp-p_h)(t/2),q_h(t/2))\\ &\quad +Ch^m t^{-(m-i)/2+1}\,\|\,p(0)\,\|_{H^i(\Omega)}\|\,\psi\,\|\\ &\leq Ch^m t^{-(m-i)/2+1}\,\|\,p(0)\,\|_{H^i(\Omega)}\|\,\psi\,\|\,.\end{aligned}$$

This proves (43).

Now we are ready to formulate and prove the main result in this section.

Theorem 2 *Let $(p,\boldsymbol{u})\in W\times\boldsymbol{V}$ and $(p_h,\boldsymbol{u}_h)\in W_h\times\boldsymbol{V}_h$ be the solutions of problems* (2) *and* (3)*, respectively. If $f=0$, $p_h(0)=Q_hp(0)$, and $p(0)\in\dot{H}^i(\Omega)$, then,*

$$\|\,(\boldsymbol{u}-\boldsymbol{u}_h)(t)\,\|\leq Ch^m t^{-(m+1-i)/2}\,\|\,p(0)\,\|_{H^i(\Omega)}, \tag{49}$$

where $0\leq i\leq m\leq r+1$.

Proof : Taking in the error equations (20) and (21), $\psi_h=Q_hp-p_h$ and $\boldsymbol{\varphi}_h=\mathbf{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h$, we get

$$\alpha_0\,\|\,\mathbf{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h\,\|^2\leq(\alpha(\mathbf{\Pi}_h\boldsymbol{u}-\boldsymbol{u}),\mathbf{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h)-(Q_hp-p_h,p_{h,s}-p_s),$$

which yield

$$\|\,\mathbf{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h\,\|^2\leq C\,\|\,\mathbf{\Pi}_h\boldsymbol{u}-\boldsymbol{u}\,\|^2+C\,\|\,Q_hp-p_h\,\|\|\,p_{h,s}-p_s\,\|\,.$$

Now we apply (5), (6), (11), and (43) to deduct (49). This completes the proof. #

4 Superconvergence

The estimate obtained in the previous sections can be interpreted in the following way: the maximum rate of convergence for $p - p_h$ is $O(h^{r+1}t^{-(r+1-i)/2})$ while for $\boldsymbol{u} - \boldsymbol{u}_h$ is $O(h^{r+1}t^{r+2-i)/2})$ for initial data in $\dot{H}^i(\Omega)$. Obviously, for any fixed $t > 0$, the convergence is asymptotically $O(h^{r+1})$, due to the smoothing properties of the parabolic operator. For smooth initial data, i.e., $p(0)$ from $\dot{H}^{r+1}(\Omega)$, the error for p is asymptotically $O(h^{r+1})$ for any t. It should be pointed out that the estimates (35) and (49) cover the whole range of smoothness of the initial data, including the worst case of data only in $L^2(\Omega)$.

The estimates (35) and (49) are of optimal type and cannot be improved in terms of the norms involved even if the solutions are smoother. However, in the case of a smooth solution one can select special points or postprocess the finite element solution in order to obtain higher order convergence. Such estimates, called superconvergence estimates, have been obtained for the standard Galerkin method for parabolic problems by Thomée, Xu and Zhang [26].

For Raviart-Thomas finite elements of order $r + 1$, which fit into the framework of this paper, Ewing and Lazarov in [13] derive $O(h^{r+2})$ error estimates for both $p - p_h$ and $\boldsymbol{u} - \boldsymbol{u}_h$ at the Guass points in the case of sufficiently smooth solution for all $t \geq 0$. Our goal in this section is to obtain superconvergent type estimates for more realistic situations, namely, for data in $\dot{H}^i(\Omega)$, for $1 \leq i \leq r + 2$.

In order to obtain estimates for $\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}$ we need an estimate for the time derivative of $(Q_hp - p_h)(t)$ in L^2-norm, which is established in the next subsection.

4.1 Superconvergence of $Q_hp - p_h$

Before we state our results, we introduce the linear functionals $\mathcal{F}(\boldsymbol{u})$ defined by

$$\mathcal{F}(\boldsymbol{u}) = \sup_{\boldsymbol{\varphi}_h \in \boldsymbol{V}_h} \frac{(\alpha(\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})), \boldsymbol{\varphi}_h)}{\|\boldsymbol{\varphi}_h\|}. \tag{50}$$

Lemma 9 *Let* $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ *and* $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ *be the solutions of problems* (2) *and* (3)*, respectively. If* $p_h(0) = Q_hp(0)$*, then*

$$\|(Q_hp - p_h)(t)\|^2 + \int_0^t \|\boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h\|^2 \; ds \leq C \int_0^t \|\mathcal{F}(\boldsymbol{u})\|^2 \; ds. \tag{51}$$

Proof : The estimate (51) follows, immediately, from (22). #

In order to use the duality argument for the error estimates discussed in Theorem 3, we need estimates for $Q_hp - p_h$ in negative Sobolev norm. These estimates are obtained in the following lemma.

Lemma 10 *Let $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ and $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ be the solutions of problems* (2) *and* (3), *respectively. If $f = 0$, $p(0) \in \dot{H}^{r+2}(\Omega)$ and $p_h(0) = Q_h p(0)$, then*

$$\| (Q_h p - p_h)(t) \|_{\dot{H}_t^{-j}(\Omega)} \leq Ch^{r+2+j} \, \| p(0) \|_{H^{r+2}(\Omega)}, \quad 0 \leq j \leq r-1, \tag{52}$$

and

$$\| (Q_h p - p_h)(t) \|_{\dot{H}_t^{-r}(\Omega)} \leq Ch^{2r+2} \, | \log h \,| \, \| p(0) \|_{H^{r+2}(\Omega)} \, . \tag{53}$$

In addition, if $\mathcal{F}(\boldsymbol{u}) \leq Ch^{r+2} \, \| \boldsymbol{u} \|_{H^{r+2}(\Omega)}$, then

$$\| (Q_h p - p_h)(t) \|_{\dot{H}_t^{-r}(\Omega)} \leq Ch^{2r+2} \, \| p(0) \|_{H^{r+2}(\Omega)} \, . \tag{54}$$

Proof : The proof follows the argument and the notations of Lemma 4. Our starting point here is also the identity (26). For any $\psi \in \dot{H}_t^{r+2}(\Omega)$, we split the estimate of the right hand side in the following way. First, for any $0 \leq j \leq r+1$, using (49) for $k = 0$, we have

$$\begin{aligned} \int_{t-h^2}^{t} (\alpha(\boldsymbol{v}_h - \boldsymbol{v}), \boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}) \, ds &\leq Ch^{r+1+j} \int_{t-h^2}^{t} (t-s)^{-1/2} ds \cdot \| p(0) \|_{H^{r+2}(\Omega)} \| \psi \|_{H^j(\Omega)} \\ &\leq Ch^{r+2+j} \, \| p(0) \|_{H^{r+2}(\Omega)} \| \psi \|_{H^j(\Omega)}, \end{aligned} \tag{55}$$

$$\begin{aligned} \int_0^t (p_s - Q_h p_s, q - Q_h q) ds &\leq Ch^{r+2+j} \int_0^t \| p_s \|_{H^{r+1}(\Omega)} \| q \|_{H^{j+1}(\Omega)} \, ds \\ &\leq Ch^{r+2+j} \int_0^t s^{-1/2} (t-s)^{-1/2} ds \, \| p(0) \|_{H^{r+2}(\Omega)} \| \psi \|_{H^j(\Omega)} \\ &\leq Ch^{r+2+j} \, \| p(0) \|_{H^{r+2}(\Omega)} \| \psi \|_{H^j(\Omega)} \, . \end{aligned} \tag{56}$$

Next, the remaining term $\int_{t-h^2}^{t} (\alpha(\boldsymbol{v}_h - \boldsymbol{v}), \boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}) \, ds$ will be estimated in the following two cases for $0 \leq j \leq r-1$ and $j = r$. When $0 \leq j \leq r-1$, we have

$$\begin{aligned} \int_0^{t-h^2} (\alpha(\boldsymbol{v}_h - \boldsymbol{v}), \boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}) \, ds &\leq Ch^{r+3+j} \int_0^{t-h^2} (t-s)^{-3/2} ds \cdot \| p(0) \|_{H^{r+2}(\Omega)} \| \psi \|_{H^j(\Omega)} \\ &\leq Ch^{r+2+j} \, \| p(0) \|_{H^{r+2}(\Omega)} \| \psi \|_{H^j(\Omega)} \, . \end{aligned}$$

Consequently, by (26), we obtain, for $0 \leq j \leq r-1$, that

$$(Q_h p - p_h)(t), \psi) \leq Ch^{r+2+j} \, \| p(0) \|_{H^{r+2}(\Omega)} \| \psi \|_{H^j(\Omega)},$$

which proves (52). Furthermore, for $j = r$, we have

$$\begin{aligned} \int_0^{t-h^2} (\alpha(\boldsymbol{v}_h - \boldsymbol{v}), \boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}) \, ds &\leq Ch^{2r+2} \int_0^{t-h^2} (t-s)^{-1} ds \cdot \| \psi \|_{H^r(\Omega)} \| p(0) \|_{H^{r+2}(\Omega)} \\ &\leq Ch^{2r+2} \, | \log h \,| \, \| \psi \|_{H^r(\Omega)} \| p(0) \|_{H^{r+2}(\Omega)} \, . \end{aligned}$$

This, together with (55) and (56), results in

$$(Q_h p - p_h)(t), \psi) \leq Ch^{2r+2} \, | \log h \,| \, \| \psi \|_{H^r(\Omega)} \| p(0) \|_{H^{r+2}(\Omega)},$$

which proves (53). Finally, (54) is an immediate corollary of Lemma 9 and the assumption for $\mathcal{F}(\boldsymbol{u})$. This completes the proof. #

Now, we are ready to prove the main results of this section, namely, estimates of $Q_h p - p_h$ for problems with nonsmooth initial data.

Theorem 3 *Let* $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ *and* $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ *be the solutions of problems* (2) *and* (3), *respectively. If* $f = 0, p(0) \in \dot{H}^i(\Omega)$, *and* $p_h(0) = Q_h p(0)$, *then, for* $r \geq 1$,

$$\| (Q_h p - p_h)(t) \| \leq C h^{r+2} t^{-(r+2-i)/2} \| p(0) \|_{H^i(\Omega)}, \quad 1 \leq i \leq r+2. \tag{57}$$

Further, for $r = 0$,

$$\| (Q_h p - p_h)(t) \| \leq C h^2 \, | \log h \, | \, t^{-(2-i)/2} \| p(0) \|_{H^i(\Omega)}, \quad 1 \leq i \leq 2, \tag{58}$$

and if $\mathcal{F}(\boldsymbol{u}) \leq C h^2 \| \boldsymbol{u} \|_{H^2(\Omega)}$, *then,*

$$\| (Q_h p - p_h)(t) \| \leq C h^2 t^{-(2-i)/2} \| p(0) \|_{H^i(\Omega)}, \quad 1 \leq i \leq 2. \tag{59}$$

Proof : First, we note that, for $i = r+2$, the estimate (57) is a particular case of (52) for $j = 0$. Thus, we have to consider the case $1 \leq i \leq r+1$. We are going to follow the arguments and notations of Theorem 2. We rewrite the identity (36) as follows:

$$\begin{aligned} ((Q_h p - p_h)(t), \psi) \;\; = \;\; & (p(0) - Q_h p(0), E^*(t,0)\psi - Q_h E^*(t,0)\psi) \\ & + (p(0), Q_h E^*(t,t/2) E^*(t/2,0)\psi - E_h^*(t,t/2) Q_h E^*(t/2,0)\psi) \\ & + (\psi, Q_h E(t,t/2) p(t/2) - E_h(t,t/2) Q_h p(t/2)) \\ & + (p_h(t/2) - p(t/2), E^*(t,t/2)\psi - E_h^*(t,t/2) Q_h \psi). \end{aligned} \tag{60}$$

Thus, using Lemma 4 and Lemma 10, we have

$$\begin{aligned} ((Q_h p - p_h)(t), \psi) \;\; \leq \;\; & C h^i \| p(0) \|_{H^i(\Omega)} \cdot h^{r+2-i} \| E^*(t,0)\psi \|_{H^{r+2-i}(\Omega)} \\ & + \| p(0) \|_{H^i(\Omega)} \cdot C h^{i+r+2-i} \| E^*(t,t/2)\psi \|_{H^{r+2-i}(\Omega)} \\ & + \| \psi \| \cdot C h^{r+2} \, | \log h \, |^{\delta_{0r}} \| p(t/2) \|_{H^{r+2}(\Omega)} \\ & + C h^i \| p(0) \|_{H^i(\Omega)} \cdot C h^{r+2-i} t^{-(r+2-i)/2} \| \psi \| \\ \leq \;\; & C h^{r+2} t^{-(r+2-i)/2} \| \psi \| \| p(0) \|_{H^i(\Omega)}, \end{aligned}$$

which, with $\delta_{0r} = 0$ if $r \neq 0$ or $\mathcal{F}(\boldsymbol{u}) \leq C h^2 \| \boldsymbol{u} \|_{\boldsymbol{H}^2(\Omega)}$, $\delta_{0r} = 1$ if $r = 0$, proves the theorem. #

4.2 Superconvergence of $\Pi_h u - u_h$

Corresponding to Lemma 7, we have

Lemma 11 *Let* $(p,\boldsymbol{u}) \in W \times \boldsymbol{V}$ *and* $(p_h,\boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ *be the solutions of problems* (2) *and* (3)*, respectively. If* $p_h(0) = Q_h p(0)$*, then,*

$$
\begin{aligned}
& t^{\gamma} \, \|(Q_h p - p_h)(t)\|^2 + \int_0^t s^{\gamma} \, \|\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h\|^2 \; ds \\
\leq \; & C \int_0^t s^{\gamma-1} \, \|Q_h p - p_h\|^2 \, ds + C \int_0^t s^{\gamma} \, |\mathcal{F}(\boldsymbol{u})|^2 \, ds,
\end{aligned}
\tag{61}
$$

where $0 \leq i \leq m \leq r+1$*,* $\gamma > 1$*.*

Proof : We can complete the proof of Lemma 11 with the same way as in Lemma 7. We omit the details. #

The main result of this section that is formulated in the following theorem.

Theorem 4 *Let* $(p,\boldsymbol{u}) \in W \times \boldsymbol{V}$ *and* $(p_h,\boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ *be the solutions of problems* (2) *and* (3)*, respectively. If* $f = 0$*,* $p(0) \in \dot{H}^i(\Omega)$*,* $p_h(0) = Q_h p(0)$*, and*

$$
\mathcal{F}(\boldsymbol{u}) \leq C h^{r+1+\beta} \, \|\boldsymbol{u}\|_{\boldsymbol{H}^{r+2}(\Omega)}, \quad \textit{for some } \beta \in (0,1],
\tag{62}
$$

then

$$
\|(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)(t)\| \leq C h^{r+1+\beta} t^{-(r+3-i)/2} \, \|p(0)\|_{H^i(\Omega)},
\tag{63}
$$

where $1 \leq i \leq r+2$*.*

Proof : The proof is similar to that of Theorem 2. First, we prove

$$
\|(Q_h p - p_h)'(t)\| \leq C h^{r+1+\beta} t^{-(r+4-i)/2} \, \|p(0)\|_{H^i(\Omega)} \, .
\tag{64}
$$

The starting point is the identity (44). Examining the terms involved in the right hand side of (44) and using Lemma 11, we have, for $s \in [t/2, t]$ and sufficient large γ, that

$$
\begin{aligned}
& \left| \int_s^t (\alpha'(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{v}_h) \, d\xi \right| \\
\leq \; & C t^{-\gamma/2} \left(\int_{t/2}^t \xi^{\gamma} \, \|(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h)\|^2 \, d\xi \right)^{1/2} \left(\int_{t/2}^t \|\boldsymbol{v}_h\|^2 \, d\xi \right)^{1/2} \\
\leq \; & C t^{-\gamma/2} \, \|\psi\| \left(\int_0^t \left(s^{\gamma-1} \, \|(Q_h p - p_h)\|^2 \; ds + s^{\gamma} \, |\mathcal{F}(\boldsymbol{u})|^2 \right) ds \right)^{1/2} .
\end{aligned}
\tag{65}
$$

Applying to the right hand side of this inequality the Theorem 3 and the assumption for $\mathcal{F}(\boldsymbol{u})$, we get

$$
\left| \int_s^t \left(\alpha' (\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{v}_h \right) d\xi \right| \leq C h^{r+1+\beta} t^{-(r+2-i)/2-1} \, \|p(0)\|_{H^i(\Omega)} \|\psi\| \, .
\tag{66}
$$

Multiplying (44) by $t-s$, and using (65) and (66), we obtain, for $s \in [t/2, t]$,

$$\begin{aligned}(t-s)\left((Q_h p - p_h)'(t), \psi\right) &\leq \frac{d}{ds}\left[(t-s)\left((Q_h p - p_h)(s), q_h(s)\right)\right] \\ &\quad + Ch^{r+2} t^{-(r+2-i)/2} \parallel p(0) \parallel_{H^i(\Omega)} \parallel \psi \parallel,\end{aligned}$$

which after integration with respect to s over the interval $[t/2, t]$, leads us to

$$\begin{aligned}\frac{1}{2}\left(\frac{t}{2}\right)^2 \left((Q_h p - p_h)'(t), \psi\right) &\leq -\frac{t}{2}\left((Q_h p - p_h)(t/2), q_h(t/2)\right) \\ &\quad + Ch^{r+2} t^{-(r+2-i)/2+1} \parallel p(0) \parallel_{H^i(\Omega)} \parallel \psi \parallel \\ &\leq Ch^{r+2} t^{-(r+2-i)/2+1} \parallel p(0) \parallel_{H^i(\Omega)} \parallel \psi \parallel .\end{aligned}$$

This proves (64).

To prove (63) we take in the error equations (20) and (21), $\psi_h = Q_h p - p_h$ and $\boldsymbol{\varphi}_h = \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h$, and get

$$\alpha_0 \parallel \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h \parallel^2 \leq (\alpha(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h) - (Q_h p - p_h, p_{h,s} - p_s).$$

Obviously, by the definition and the assumption for $\mathcal{F}(\boldsymbol{u})$, we have

$$(\alpha(\boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h) \leq \mathcal{F}(\boldsymbol{u}) \parallel \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h \parallel \leq Ch^{r+1+\beta} \parallel \boldsymbol{u} \parallel_{H^{r+2}(\Omega)} \parallel \boldsymbol{\Pi}_h \boldsymbol{u} - \boldsymbol{u}_h \parallel .$$

The second term is estimated by Schwarz inequality and the estimate (64) and Lemma 10. This completes the proof. #

4.3 Superconvergent recovery of the solution at the Gauss points for Riviart-Thomas finite element

Now we shall present one possible way of using the results of Theorems 3 and 4 for superconvergent recovery of the solution p and $\boldsymbol{u}$ from their finite element approximations p_h and $\boldsymbol{u}_h$ for problems on smooth domains. We assume that T_h consists of rectangles with sides parallel to the coordinate axis with possible exception of triangles and rectangles with one curved side that is part of the boundary of Ω.

The construction of the spaces $\boldsymbol{V}_h$ and W_h uses Raviart-Thomas elements for rectangles and Douglas-Roberts curved elements near the boundary. All rectangular elements form the set T_h^0 and the remaining elements are in the set T_h^1. Note that we require that the elements in T_h^1 are included in a strip Ω_1 around the boundary with width $O(h)$. According to [10], the spaces $\boldsymbol{V}_h$ and W_h defined in this way satisfy the properties (5) and (6) and the projection operators are defined element by element. For the functional $\mathcal{F}(\boldsymbol{u})$ we have

$$(\alpha(\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}), \boldsymbol{\varphi}) = \sum_{e \in T_h^0} \int_e \alpha(\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}) \boldsymbol{\varphi} dx + \sum_{e \in T_h^1} \int_e \alpha(\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u}) \boldsymbol{\varphi} dx = I_0 + I_1. \tag{67}$$

It has been shown by Ewing and Lazarov in [13] that for rectangular elements $e \in T_h^0$

$$\int_e \alpha(\boldsymbol{u} - \mathbf{\Pi}_h \boldsymbol{u})\boldsymbol{\varphi} dx \le Ch^{r+2} \parallel \boldsymbol{u} \parallel_{\boldsymbol{H}^{r+2}(e)} \parallel \boldsymbol{\varphi} \parallel .$$

Therefore, the first term I_0 is estimated by

$$| I_0 | \le Ch^{r+2} \parallel \boldsymbol{u} \parallel_{\boldsymbol{H}^{r+2}(\Omega)} \parallel \boldsymbol{\varphi} \parallel .$$

Note that the second term involves only elements in a strip of width $O(h)$ near the boundary $\partial\Omega$. Using the well known inequality

$$\parallel \boldsymbol{u} \parallel_{\boldsymbol{H}^{r+1}(\Omega_1)} \le Ch^{1/2} \parallel \boldsymbol{u} \parallel_{\boldsymbol{H}^{r+2}(\Omega)} .$$

and the approximation property of finite element space $\boldsymbol{V}_h$, we have

$$| I_1 | \le Ch^{r+1} \sum_{e \in T_h^1} \parallel \boldsymbol{u} \parallel_{\boldsymbol{H}^{r+1}(e)} \parallel \boldsymbol{\varphi} \parallel_{L^2(e)} \le Ch^{r+3/2} \parallel \boldsymbol{u} \parallel_{\boldsymbol{H}^{r+2}(\Omega)} \parallel \boldsymbol{\varphi} \parallel .$$

From these estimates follows the estimate (62) for $\mathcal{F}(\boldsymbol{u})$ with $\beta = 1/2$.

Next, we introduce some discrete seminorms in W and $\boldsymbol{V}$. Let $\hat{g}_i$, $i = 1, ..., r+1$ be the Gauss points in the interval $[-1, 1]$, i.e., $L_{r+1}(\hat{g}_i) = 0$, where L_{r+1} is the Legendre polynomial of degree $r+1$ orthogonal on the interval $[-1, 1]$. Each rectangular finite element $e \in T_h^0$ can be transformed by an affine mapping F to the reference element $\hat{e} = [-1, 1] \times [-1, 1]$. Then, the inverse mapping F^{-1} introduces in e the set of points $G(e) = \{F^{-1}(\hat{g}_i, \hat{g}_j), i, j = 1, \cdots, r+1\}$. The seminnorms in W and $\boldsymbol{V}$ are defined by

$$|||w|||= \left(\sum_{e \in T_h^0} \sum_{x \in G(e)} h^2 w^2(x) \right)^{1/2}, \qquad |||\boldsymbol{v}|||=|||v_1||| + |||v_2|||,$$

where $\boldsymbol{v} = (v_1, v_2)$. It is easy to see that there is a constant $C > 0$ such that

$$|||w_h||| \le C \parallel w_h \parallel, \qquad \text{for all } w_h \in W_h$$

and

$$|||\boldsymbol{v}_h||| \le C \parallel \boldsymbol{v}_h \parallel, \qquad \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Now, we estimate the errors for $p - p_h$ and $\boldsymbol{u} - \boldsymbol{u}_h$ in the discrete seminorms defined above.

Theorem 5 *Let $(p, \boldsymbol{u}) \in W \times \boldsymbol{V}$ and $(p_h, \boldsymbol{u}_h) \in W_h \times \boldsymbol{V}_h$ be the solutions of problems (2) and (3), respectively. If $f = 0, p(0) \in \dot{H}^i(\Omega)$ and $p_h(0) = Q_h p(0)$, then*

$$|||(p - p_h)(t)||| \le Ch^{r+2} \, | \log h |^{\bar{r}} \, t^{-(r+2-i)/2} \parallel p(0) \parallel_{H^i(\Omega)}, \tag{68}$$

$$|||(\boldsymbol{u} - \boldsymbol{u}_h)(t)||| \le Ch^{r+3/2} t^{-(r+3-i)/2} \parallel p(0) \parallel_{H^i(\Omega)}, \tag{69}$$

where $1 \le i \le r+2$ and $\bar{r} = 1$ if $r = 0$ and $\bar{r} = 0$ if $r \ne 0$.

Proof : From the discussion presented above, we have

$$\begin{aligned} |||p - p_h||| &\leq |||p - Q_h p||| + |||Q_h p - p_h||| \\ &\leq |||p - Q_h p||| + C \, \| Q_h p - p_h \| \, . \end{aligned} \tag{70}$$

The second term in the right hand side of (70) has been already estimated in Theorem (3). Now, we estimate the first term. It is easy to show (see for details [13]) that for each element $e \in T_h^0$,

$$\sum_{x \in G(e)} h^2 (p - Q_h p)^2 (x,t) = 0, \qquad \text{if } p \in P_{r+1}(e).$$

Thus, by the Bramble-Hilbert lemma argument, one obtains

$$\left| \sum_{x \in G(e)} h^2 (p - Q_h p)^2 (x,t) \right|^{1/2} \leq C h^{r+2} \, \| p(t) \|_{H^{r+2}(e)} \, . \tag{71}$$

We sum (71) over $e \in T_h^0$ and use the estimate (11) to obtain

$$|||p - Q_h p||| \leq C h^{r+2} t^{-(r+2-i)/2} \, \| p(0) \|_{H^i(\Omega)}, \qquad 1 \leq i \leq r+2.$$

This proves (68). The estimate (69) is proven similarly. #

Obviously, we have full superconvergence for the pressure at the Gaussian points in all rectangular elements. This is due to the fact that the superconvergence estimate in Theorem 3 are derived for any meshes without any additional conditions. In contrast, in the estimate for $\boldsymbol{u} - \boldsymbol{u}_h$ we have lost 1/2 order. This is due to the fact that we use irregular elements near the boundary.

The derived superconvergence estimates represent only an example of using the theory developed above. It is natural to expect that one may get full order superconvergence in the velocity in the interior of the domain. This has been already observed in the computation of Ewing and Wheeler [15]. However, interior error estimates require different techniques and are not in the scope of this paper.

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