MULTIPHASE FLOW SIMULATION WITH VARIOUS BOUNDARY CONDITIONS

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Multiphase flow simulation with various nonhomogeneous boundary conditions in groundwater hydrology and petroleum engineering is considered. The phase flow equations are given in a fractional flow formulation, i.e., in terms of a saturation and a global pressure. It is shown that most commonly used boundary conditions for groundwater hydrology and petroleum engineering problems can be incorporated into the pressure-saturation formulation.

INTRODUCTION

In petroleum reservoir simulation the governing equations that describe fluid flow are usually written in a fractional flow formulation, i.e., in terms of a saturation and a global pressure [5], [9]. The main reason for this fractional flow approach is that efficient numerical methods can be devised to take advantage of many physical properties inherent in the flow equations. However, this pressure-saturation formulation has not yet achieved wide application in groundwater hydrology. In petroleum reservoirs total flux type boundary conditions are conveniently imposed and often used, but in groundwater reservoirs boundary conditions are very complicated.

In this paper multiphase flow simulation with various nonhomogeneous boundary conditions in groundwater hydrology and petroleum engineering is considered. We show that most commonly encountered boundary conditions in groundwater reservoirs can be incorporated in the fractional flow formulation. A numerical method based on use of a mixed finite element method for the global pressure and a standard Galerkin method for the saturation is presented.

DIFFERENTIAL EQUATIONS

Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be a porous medium. The usual equations describing two-phase flow

in Ω are given by the mass balance equation and Darcy's law for each of the fluid phases

$$(1.1) \qquad \frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha v_\alpha) = f_\alpha, \quad x \in \Omega, \, t > 0,$$

$$(1.2) \qquad v_\alpha = -\frac{kk_{r\alpha}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g), \qquad x \in \Omega, \, t > 0,$$

where $\alpha = w$ denotes the wetting phase (e.g. water), $\alpha = a$ indicates the nonwetting phase (e.g. air or oil), ϕ and k are the porosity and absolute permeability of the porous system, ρ_α, s_α, p_α, v_α, and μ_α are the density, saturation, pressure, volumetric velocity, and viscosity of the α-phase, f_α is the source/sink term, $k_{r\alpha}$ is the relative permeability of the α-phase, and g is the gravitational, downward-pointing, constant vector. Impose the customary property that the fluid fills the volume:

$$(1.3) \qquad s_a + s_w = 1.$$

Also, define the capillary pressure function p_c by

$$(1.4) \qquad p_c(s_w) = p_a - p_w.$$

For notational convenience, introduce the phase mobility function

$$(1.5) \qquad \lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}, \quad \alpha = w, a.$$

Now, substitute (1.3)–(1.5) into (1.1) and (1.2) to obtain the usual two-pressure equation formulation. The most commonly encountered boundary conditions for the two-pressure equations are of first-type, second-type, third-type, and "well" type [1], [5]. Let $\partial\Omega$ be a set of three disjoint regions Γ_i, $i = 1, 2, 3$, and let $\Gamma_3 = \cup_j \Gamma_{3,j}$ where each $\Gamma_{3,j}$ is connected. Then we consider for $\alpha = w, a$ and $s = s_w$,

$$(1.6) \qquad p_\alpha = p_{\alpha D}(x, t), \qquad\qquad\qquad x \in \Gamma_1, \, t > 0,$$

$$(1.7) \qquad v_\alpha \cdot \nu + b_\alpha(x, t, s)p_\alpha = g_\alpha(x, t, s), \quad x \in \Gamma_2, \, t > 0,$$

$$(1.8a) \qquad \int_{\Gamma_{3,j}} (v_w + v_a) \cdot \nu = g_j(t), \qquad\qquad x \in \Gamma_{3,j}, \, t > 0,$$

$$(1.8b) \qquad p_\alpha = p_{\alpha D}(x, t) + d_j(t), \qquad\qquad x \in \Gamma_{3,j}, \, t > 0,$$

where $p_{\alpha D}$, b_α, g_α, and g_j are given functions, d_j is an arbitrary scaling constant, and ν is the outer unit normal to $\partial\Omega$. The initial condition is given as

$$(1.9) \qquad s_w(\cdot, 0) = s_w^0, \quad x \in \Omega.$$

The model is now completed.

Pressure-saturation formulation

To devise our numerical method, as mentioned in the introduction we rewrite (1.1) and (1.2) in a pressure-saturation formulation. For this, define the global pressure [5]

$$(1.10) \qquad p = \frac{1}{2}(p_w + p_a) + \frac{1}{2} \int_{s_c}^{s} \frac{\lambda_a - \lambda_w}{\lambda} \frac{dp_c}{d\xi} d\xi$$

$$= p_w + \int_{0}^{p_c(s)} \left(\frac{\lambda_a}{\lambda}\right) \left(p_c^{-1}(\xi)\right) d\xi,$$

where $\lambda = \lambda_w + \lambda_a$ and $p_c(s_c) = 0$, and the total velocity

$$(1.11) \qquad v = -k\lambda \left(\nabla p - G_\lambda\right),$$

where

$$G_\lambda = \frac{\lambda_w \rho_w + \lambda_a \rho_a}{\lambda} g.$$

Then it can be easily seen that

$$(1.12a) \qquad v_w = q_w v + k\lambda_a q_w \nabla p_c - k\lambda_a q_w \delta\rho g,$$

$$(1.12b) \qquad v_a = q_a v - k\lambda_w q_a \nabla p_c + k\lambda_w q_a \delta\rho g,$$

where $q_\alpha = \lambda_\alpha/\lambda$, $\alpha = w, a$, and $\delta\rho = \rho_a - \rho_w$. Consequently,

$$(1.13) \qquad v = v_w + v_a.$$

Add (1.1) with $\alpha = w$ and $\alpha = a$ to give the pressure equation

$$(1.14) \qquad \nabla \cdot v = -\frac{\partial \phi}{\partial t} - \sum_{\alpha=w}^{a} \frac{1}{\rho_\alpha} \left(\phi s_\alpha \frac{\partial \rho_\alpha}{\partial t} + v_\alpha \cdot \nabla \rho_\alpha - f_\alpha\right).$$

Substitute (1.12a) into (1.1) with $\alpha = w$ to obtain the saturation equation

$$(1.15) \qquad \phi \frac{\partial s_w}{\partial t} + \nabla \cdot \left(q_w v + k\lambda_a q_w (\nabla p_c - \delta\rho g)\right)$$

$$= -s_w \frac{\partial \phi}{\partial t} - \frac{1}{\rho_w} \left(\phi s_w \frac{\partial \rho_w}{\partial t} + v_w \cdot \nabla \rho_w - f_w\right).$$

The capillary diffusion term $D(s)$ in this saturation equation is clearly defined by

$$D(s) = -k\lambda_a q_w \frac{dp_c}{ds}.$$

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We now have the pressure-saturation equations (1.11), (1.14), and (1.15). Let $\Gamma_{p,i} = \Gamma_i$, $i = 1, 2, 3$, $\Gamma_{s,2} = \Gamma_2$, and $\Gamma_{s,1} = \Gamma_1 \cup \Gamma_3$. Then the boundary conditions for the pressure-saturation equations become

$$(1.16) \qquad p = p_D(x,t), \qquad\qquad\qquad\qquad\qquad x \in \Gamma_{p,1},\ t > 0,$$

$$(1.17) \qquad v \cdot \nu + b(x,t,s)p = G(x,t,s), \qquad\qquad x \in \Gamma_{p,2},\ t > 0,$$

$$(1.18\text{a}) \qquad \int_{\Gamma_{p,3,j}} v \cdot \nu = g_j(t), \qquad\qquad\qquad\qquad x \in \Gamma_{p,3,j},\ t > 0,$$

$$(1.18\text{b}) \qquad p = p_D(x,t) + d_j(t), \qquad\qquad\qquad x \in \Gamma_{p,3,j},\ t > 0,$$

$$(1.19) \qquad s = s_D(x,t), \qquad\qquad\qquad\qquad\qquad x \in \Gamma_{s,1},\ t > 0,$$

$$(1.20) \qquad (q_w v + k\lambda_a q_w(\nabla p_c - \delta\rho g)) \cdot \nu$$
$$+ b_w(x,t,s)p = G_w(x,t,s), \quad x \in \Gamma_{s,2},\ t > 0,$$

where p_D and s_D are the transforms of p_{wD} and p_{aD} by (1.10) and (1.4), and

$$b = b_w + b_a,$$

$$G = g_w + g_a - b_a p_c + b \int_0^{p_c(s)} q_a\left(p_c^{-1}(\xi)\right) d\xi,$$

$$G_w = g_w + b_w \int_0^{p_c(s)} q_a\left(p_c^{-1}(\xi)\right) d\xi.$$

The initial condition is the same as in (1.9).

Petroleum reservoirs

The flow of two incompressible fluids (e.g. water and oil) in a porous medium Ω has been extensively studied by petroleum engineers for many decades. In this case the pressure-saturation equations (1.11), (1.14), and (1.15) reduce to the following simplified equations:

$$(1.21) \qquad \nabla \cdot v = -\frac{\partial \phi}{\partial t} + f, \qquad\qquad\qquad\qquad x \in \Omega,\ t > 0,$$

$$(1.22) \qquad v = -k\lambda\left(\nabla p - G_\lambda\right), \qquad\qquad\qquad x \in \Omega,\ t > 0,$$

$$(1.23) \qquad \phi\frac{\partial s_w}{\partial t} + \nabla \cdot (q_w v + k\lambda_o q_w(\nabla p_c - \delta\rho g)) = -s_w\frac{\partial \phi}{\partial t} + \tilde{f}_w, \quad x \in \Omega,\ t > 0,$$

where $f = f_w/\rho_w + f_o/\rho_o$ and $\tilde{f}_w = f_w/\rho_w$. We remark that $\frac{\partial \phi}{\partial t}$ is quite small, and is usually neglected. Typical examples of the relative permeability functions $k_{r\alpha}$, $\alpha = w, o$, the capillary pressure function p_c, the fractional flow function q_w, and the capillary diffusion function D for an oil-water system are plotted in Figure 1.

Groundwater hydrology

We now consider an air-water system where the water is assumed to be incompressible, but the air is supposed to be compressible. Furthermore, the air density is assumed to be a function of air pressure. Then we see from (1.10) that

$$\frac{1}{\rho_a}\frac{\partial \rho_a}{\partial t} = c_a \left(\frac{\partial p}{\partial t} + q_w \frac{\partial p_c}{\partial t} \right),$$

$$\frac{1}{\rho_a}\nabla \rho_a = c_a \left(\nabla p + q_w \nabla p_c \right),$$

where compressibility c_a is defined by

$$(1.24) \qquad\qquad c_a = \frac{1}{\rho_a}\frac{d\rho_a}{dp_a}.$$

Apply these equations in (1.11), (1.14), and (1.15) to obtain

$$(1.25) \quad s_a c_a \frac{dp}{dt} + \nabla \cdot v = -\frac{\partial \phi}{\partial t} - s_a c_a q_w \frac{dp_c}{dt} + f, \qquad\qquad x \in \Omega,\ t > 0,$$

$$(1.26) \quad v = -k\lambda \left(\nabla p - G_\lambda \right), \qquad\qquad x \in \Omega,\ t > 0,$$

$$(1.27) \quad \phi \frac{\partial s_w}{\partial t} + \nabla \cdot (q_w v + k\lambda_a q_w (\nabla p_c - \delta \rho g)) = -s_w \frac{\partial \phi}{\partial t} + \tilde{f}_w, \quad x \in \Omega,\ t > 0,$$

where

$$\frac{d}{dt} = \phi \frac{\partial}{\partial t} + \frac{v_a}{s_a} \cdot \nabla.$$

Note that the pressure equation is a parabolic equation in the present situation. Thus we need the initial condition

$$p(\cdot, 0) = p^0, \quad x \in \Omega.$$

An example of the air density function is given by the relation [13]

$$\rho_a = \rho_{0a} \left(1 + \frac{p_a}{p_{0a}} \right),$$

where ρ_{0a} is the density of the air phase at the pressure p_{0a}. Typical examples of the relative permeability functions $k_{r\alpha}$, $\alpha = w, a$, the capillary pressure function p_c, the fractional flow function q_w, and the capillary diffusion function D for an air-water system are plotted in Figure 2. The initial and boundary conditions in (1.9) and (1.16)–(1.20) remain the same in the two situations above.

FINITE ELEMENT METHOD

We now develop a finite element approximation procedure for numerically solving (1.21)–(1.23) and (1.25)–(1.27). We only consider the latter case, i.e., an air-water system; the

Fig. 1. a) Relative permeability
b) fractional flow function c) capillary pressure
d) capillary diffusion function for an oil-water system.

former case is simpler. Let $(\cdot, \cdot)_R$ denote the $L^2(R)$-inner product (we omit R if $R = \Omega$), and set $S = \{\psi \in H^1(\Omega) : \psi|_{\Gamma_{s,1}} = 0\}$. For $0 < h_p < 1$ and $0 < h < 1$, let T_{h_p} and T_h be quasiregular partitions of Ω. For each $R \in T_{h_p}$, let $V_h(R) \times W_h(R)$ denote some standard mixed finite element space for second order elliptic problems (see, e.g., [2], [3], [4], [6], [10],

and [12]). Then we define

$$V_h = \{\xi \in (L^2(\Omega))^d : \xi|_R \in V_h(R) \text{ for each } R \in T_{h_p}\},$$
$$W_h = \{\psi \in L^2(\Omega) : \psi|_R \in W_h(R) \text{ for each } R \in T_{h_p}\},$$
$$L_{h,\pi} = \left\{ r \in L^2\left(\bigcup_{e\in\partial T_{h_p}} e \right) : r|_e \in V_h \cdot \nu|_e \text{ for each } e \in \partial T_{h_p}; \right.$$
$$(r - \pi, r_1)_e = 0, \ r_1 \in V_h \cdot \nu|_e, \ \forall e \in \Gamma_{p,1};$$
$$\left. (r - \pi - d_j, r_1)_e = 0, \ r_1 \in V_h \cdot \nu|_e, \ \forall e \in \Gamma_{p,3,j}, \text{ for each } j \right\}.$$

Finally, let $S_h \subset S$ be a standard C^0 finite element space [7] associated with T_h. Our finite element method is formulated as follows. The mixed finite element solution of the pressure equation is $\{v_h, p_h, l_h\} : (0, \infty) \to V_h \times W_h \times L_{h,p_D}$ satisfying

$$\left((1 - s_h)c_a(s_h, p_h)\frac{dp_h}{dt}, \psi \right) + \sum_{R\in T_{h_p}} (\nabla \cdot v_h, \psi)_R$$
$$= -\left(\frac{\partial\phi}{\partial t} + (1 - s_h)c_a(s_h, p_h)\frac{dp_c(s_h)}{dt} - f(s_h, p_h), \psi \right), \quad \forall\psi \in W_h,$$
$$\left((k\lambda(s_h))^{-1}v_h, \xi \right) - \sum_{R\in T_{h_p}} \left((p_h, \nabla \cdot \xi)_R - (l_h, \xi \cdot \nu_R)_{\partial R} \right)$$
$$= (G_\lambda(s_h, p_h), \xi), \quad \forall\xi \in V_h,$$
$$\sum_{R\in T_{h_p}} (v_h \cdot \nu_R, r)_{\partial R\backslash\Gamma_{p,1}} = (G(s_h) - b(s_h)l_h, r)_{\Gamma_{p,2}} + \sum_j \frac{(g_j, r)_{\Gamma_{p,3,j}}}{|\Gamma_{p,3,j}|}, \quad \forall r \in L_{h,0},$$
$$p_h(\cdot, 0) = p_h^0, \quad x \in \Omega,$$

and the finite element solution of the saturation equation is $s_h : (0, \infty) \to S_h + s_D$ satisfying

$$\left(\phi\frac{\partial s_h}{\partial t}, \psi \right) - (q_w(s_h)v_h + k(\lambda_a q_w)(s_h)(\nabla p_c(s_h) - \delta\rho(s_h, p_h)g), \nabla\psi)$$
$$= -(G_w(s_h) - b_w(s_h)l_h, \psi)_{\Gamma_{s,2}} + \left(\tilde{f}_w - s_h\frac{\partial\phi}{\partial t}, \psi \right), \quad \forall\psi \in S_h,$$
$$s_h(\cdot, 0) = s_h^0, \quad x \in \Omega,$$

where p_h^0 and s_h^0 are some approximations in W_h and S_h of p^0 and s^0, respectively. We conclude with three remarks. First, while, for completeness, the standard finite element method is considered for the saturation equation here, due to its convection-dominatedness feature it can be solved using more efficient numerical approaches such as characteristic

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Petrov-Galerkin methods based on operator splitting [8], transport diffusion methods [11], and other characteristic based methods. Second, the Lagrange multipliers over edges or faces are here used. The reasons for this are that the linear system arising from this unconstrained mixed formulation leads to a symmetric, positive definite system for the Lagrange multipliers, which can be easily solved, and that the boundary conditions (1.16)–(1.20) can be easily incorporated in this formulation. Finally, note that we have a coupled nonlinear system for the velocity v_h, the pressure p_h, and the saturation s_h. In general, the boundary data depend on the saturation, which makes the whole system even more difficult to solve. Our future work will be concentrated on development of computer programs based on the pressure-saturation formulation for physically reasonable data and on extension of the present techniques to three-phase flow.

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Fig. 2. a) Relative permeability
b) fractional flow function c) capillary pressure
d) capillary diffusion function for an air-water system.

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