SUBSTRUCTURING PRECONDITIONING FOR FINITE ELEMENT APPROXIMATIONS OF SECOND ORDER ELLIPTIC PROBLEMS. I. NONCONFORMING LINEAR ELEMENTS FOR THE POISSON EQUATION IN A PARALLELEPIPED

R.E. Ewing[1] , *Yu. Kuznetsov*[2] , *R.D. Lazarov*[3] , *and S. Maliassov*

1. Introduction

Let Ω be a convex polyhedral domain in $\mathbb{R}^3$, $f(x) \in L^2(\Omega)$ and $A(x)$ be a sufficiently smooth three by three symmetric matrix-valued function on $\bar{\Omega}$ satisfying the uniform positive definiteness condition: there exists an $\alpha > 0$ such that

$$\alpha^{-1}\xi^T\xi \le \xi^T A(x)\xi \le \alpha\xi^T\xi\,, \qquad \forall x \in \bar{\Omega}\,, \forall \xi \in \mathbb{R}^3\,. \tag{1.1}$$

Throughout this paper, we use boldfaced letters to denote vectors in general in the space $\mathbb{R}^N$.

We consider the Dirichlet boundary value problem:

$$\begin{aligned} \mathbf{q} + A\nabla u = 0, &\qquad \text{in } \ \Omega\,, \\ \nabla \cdot \mathbf{q} = f, &\qquad \text{in } \ \Omega\,, \\ u = 0, &\qquad \text{on } \ \partial\Omega, \end{aligned} \tag{1.2}$$

where $\partial\Omega$ is the boundary of Ω. In applications of fluid flow in porous media, $u(x)$ is referred to as pressure and $\mathbf{q}$ as to Darcy velocity vector. It is well known that (1.2) has a unique solution $u(x) \in H_0^1(\Omega) \cap H^2(\Omega)$, and that the following elliptic regularity estimate holds true (cf. [14]):

$$\|u\|_{2,\Omega} \le c\|f\|_{0,\Omega}\,, \tag{1.3}$$

where c is a constant dependent only on Ω and where $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{2,\Omega}$ are the $L^2(\Omega)$ and $H^2(\Omega)$ Sobolev norms, respectively defined by

$$\|u\|_{0,\Omega} = \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}}, \qquad \|u\|_{2,\Omega} = \left(\int_\Omega \sum_{|\alpha|\le m} |\partial^\alpha u|^2 dx\right)^{\frac{1}{2}}. \tag{1.4}$$

The problem (1.2) can be discretized in various ways. Among the most popular and frequently used methods of approximation are the finite volume method, the Galerkin finite element method and the mixed finite element method. Each of these methods has its advantages and disadvantages when applied to particular

[1]R. Ewing, R. Lazarov, S. Maliassov, Istitute for Scientific Computation, Department of Mathematics, Texas A&M University, College Station, TX 77843.

[2]Institute of Numerical Mathematics, Russian Academy of Sciences, 32-a Leninski Prospect, 117334 Moscow, Russia.

[3]Institute of Mathematics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria.

engineering problems. For example, for petroleum reservoir problems in geometrically simple domains and heterogeneous media, the finite volume method has proven to be reliable, accurate and mass conserving cell-by-cell. Many engineering problems, e.g. petroleum recovery, ground-water contamination, seismoic exploration etc. need very accurate velocity (flux) determination in the presence of heterogeneities, anisotropy and large jumps in the coefficient matrix $A(x)$. More accurate approximation of the velocity can be achieved through the use of the mixed finite element method. As shown by Wiser and Wheeler in [21], the mixed finite element approximations with special quadratures on rectangular grids are equivalent to the finite volume methods and give superconvergent velocity calculations for smooth solutions. Based on that equivalence, Bramble et al. in [3] have developed efficient multigrid solution procedures for structured grids. However, in general the technique of the mixed finite element method leads to an algebraic saddle point problem that is more difficult and more expensive to solve. Although some reliable preconditioning algorithms for these saddle point problems have been proposed and studied (see, e.g. [4, 11, 17, 19]), their efficiency depends strongly on the geometry of the domain, on the coefficient matrix $A(x)$ and on the type of the finite elements used.

An alternative approach can be taken by developing hybrid methods. This approach has been studied in the pioneering work of Arnold and Brezzi [2] where the continuity of the velocity vector normal to the boundary of each element is enforced by Lagrange multipliers. In general, the Lagrange multipliers on the element boundaries turn out to be none other that the trace of the pressure $u(x)$.

Now we explain briefly the main idea of the Lagrange formulation of the mixed finite element method. Introduce the spaces

$$\mathbf{V} = H(div;\Omega) = \left\{ \mathbf{q} \in \left(L^2(\Omega)\right)^3, \nabla\cdot\mathbf{q} \in L^2(\Omega) \right\}, \qquad W = L^2(\Omega);$$

then the weak formulation of the system (1.2) is: find a pair $(\mathbf{q},\, u) \in \mathbf{V}\times W$ such that

$$(\nabla\cdot\mathbf{q},\, w) + (A^{-1}\mathbf{q},\, \mathbf{p}) - (u,\, \nabla\cdot\mathbf{p}) = (f,\, w), \qquad \forall\, w\in W, \quad \mathbf{p}\in\mathbf{V}. \tag{1.5}$$

The standard mixed finite element approximation to (1.5) reads as follows: let $\bar{\mathbf{V}}_h \times W_h \subset \mathbf{V}\times W$ be a finite element space over the partition $\mathcal{T}_T$ of Ω into tetrahedra (or over the partition $\mathcal{T}_C$ into cubes) (see Raviart-Thomas, Brezzi-Fortin [16, 6]). The requirement $\bar{\mathbf{V}}_h \subset \mathbf{V}$ implies that the normal component of the vector $\mathbf{q}$ is continious accross the interelement boundaries $\partial\mathcal{T}_T$. the construction of Arnold and Brezzi [2] is based on the idea of backing off this continuity requirement and defining the space $\mathbf{V}_h = \left\{ \mathbf{q} \in \left(L^2_h(\Omega)\right)^3 : \mathbf{q}|_T \in \bar{\mathbf{V}}_h,\ T\in\mathcal{T}_T \right\}$. In order to introduce the interelement continuity of the normal component of $\mathbf{q}$ we introduce the space of the Lagrange multipliers

$$L_h = \left\{ \lambda \in L^2(\partial\mathcal{T}_T) : \lambda|_{\partial T} \in \bar{\mathbf{V}}_h\cdot\nu \ \text{ for each } T\in\mathcal{T}_T \right\},$$

where ν is the normal to ∂T vector.

Now the approximation to (1.5) using Lagrange multipliers is formulated for the unknown triple $(\mathbf{q}_h,\, u_h,\, \lambda_h) \in \mathbf{V}_h\times W_h\times L_h$. We skip the details of the weak formulation of (1.5) over $\mathbf{V}_h\times W_h\times L_h$ refering to [Brenner, Z. Chen, Arbogast &

Chen, Arnold and Brezzi]. If the vectors $\mathbf{Q}$, $\mathbf{U}$ and $\mathbf{\Lambda}$ correspond to the representation of $\mathbf{q}_h$, u_h and λ_h with respect to the bases in $\mathbf{V}_h$, W_h and L_h, respectively, the algebraic form of this approximation is (see Brezzi & Fortin [6])

$$\begin{pmatrix} M & B & C \\ B^T & 0 & 0 \\ C^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Q} \\ \mathbf{U} \\ \mathbf{\Lambda} \end{pmatrix} = \mathbf{F}, \tag{1.6}$$

where M is a symmetric and positive definite matrix. Important feature of the matrices M and B is that they are block diagonal since the unknown nodal values of $\mathbf{q}_h$ and u_h over a given finite element T are related to the nodal values on the adjacent element only through the matrix C. Therefore, using element-by-element elimination we can reduce this system to the form

$$S\,\mathbf{\Lambda} = \mathbf{\Phi}. \tag{1.7}$$

For the description of the structure and the particular form of the Shur complement S in the case of particular finite element spaces we refer to [1, 5, 7, 8].

The important discovery of Arnold and Brezzi [2] is that the system (1.7) can be obtained also from application to (1.2) the Galerkin method with nonconforming elements. Namely in [2] it is shown that the lowest-order Raviart-Thomas mixed element approximations are equivalent to the usual P_1-nonconforming finite element approximations when the classical P_1-nonconforming space is augmented with P_3-bubbles. Such a relationship has been studied recently for a large variety of mixed finite element spaces [1, 5, 7].

This equivalence between the hybrid mixed and the nonconforming finite element methods establishes a framework for preconditioning and/or solving the algebraic problem and for postprocessing the finite element solution. Schematically this framework includes the following three steps:

(a) forming the reduced algebraic problem for the Lagrange multipliers, which is equivalent to the nonconforming problem;

(b) construction and study of efficient methods, based on multigrid, multilevel or domain decomposition, for solving or preconditioning the reduced problem;

(c) recovery of the solution $u(x)$ and the velocity $\mathbf{q}$ from the Lagrange multipliers that were already found.

The recent progress in each of the steps described above (see, e.g. [20, 18, 10]) gives us an indication that the mixed finite element method can be used as an accurate and efficient tool for solving general elliptic problems of second order in domains with complicated geometry.

The goal of this paper is to construct, study and implement efficient preconditioners for the nonconforming finite element approximations of problem (1.2) on arbitrary tetrahedral meshes.

2. Problem Formulation

We consider Ω to be a unit cube in $\mathbb{R}^3$ and $A(x) = a(x)\,I$ to be a scalar matrix. Let $\mathcal{T}_T$ be a regular partitioning of Ω into tetrahedra T with a characteristic size

$h = diam(T)$ (see [9]). Later on in Section 3 we introduce a special partitioning of Ω in order to get better algebraic properties of the matrix of the corresponding algebraic system (see Fig. 1).

We introduce the set Q_h of barycenters of all faces of the tetrahedral partition of Ω, and the set $\overset{\circ}{Q}_h$ of those barycenters that are strictly inside Ω. The Crouzeix-Raviart nonconforming finite element space V_h consists of all piecewise linear functions on $\mathcal{T}_T$ that vanish at the barycenters of the boundary faces and are continuous at the barycenters of $\overset{\circ}{Q}_h$. Note that the space V_h is not a subspace of $H_0^1(\Omega)$.

Now we define the bilinear form on V_h by

$$a_h(u,v) = \sum_{T \in \mathcal{T}_T} \int_T a(x)\, \nabla u \cdot \nabla v dx, \qquad \forall\ u, v \in V_h. \tag{2.1}$$

Thus the nonconfirming discretization of problem (2.1) is given by seeking $u_h \in V_h$ such that

$$a_h(u_h, v) = (f, v), \qquad \forall\ v \in V_h\,, \tag{2.2}$$

where (f, v) denotes the L^2-inner product of two functions.

The natural degrees of freedom of Crouzeix-Raviart nonconforming elements are the values at the barycenters of the faces of the tetrahedral elements. Denote the vector of the unknown values corresponding to a function $v_h \in V_h$ by $\mathbf{v}$ and assume that its dimension is N, i.e., $\mathbf{v} \in \mathbb{R}^N$. Note, that all unknowns on faces on the boundary with Dirichlet data are excluded.

Let $(\mathbf{u}, \mathbf{v})$ be a bilinear form defined on $\mathbb{R}^N$ by

$$(\mathbf{u}, \mathbf{v})_N = h^3 \sum_{x \in \overset{\circ}{Q}_h} u(x) v(x), \qquad u, v \in V_h. \tag{2.3}$$

It is clear that $(\cdot,\cdot)_N$ is equivalent to the L^2-inner product on V_h; i.e. there exists a constant $c > 0$ such that

$$c^{-1} \|v\|_0^2 \le \|\mathbf{v}\| \le c \|v\|_0^2, \qquad v \in V_h \tag{2.4}$$

where

$$\|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})_N^{\frac{1}{2}} \qquad \text{for } v \in V_h.$$

Then the discretization operator $A : \mathbb{R}^N \to \mathbb{R}^N$ (we shall call A sometimes global "stiffness" matrix) is defined by

$$(A\mathbf{v}, \mathbf{w})_N = a_h(u, w), \qquad u, v \in V_h. \tag{2.5}$$

Similarly, we introduce the vector $\mathbf{F}$ as

$$(f, v) = (\mathbf{F}, \mathbf{v})_N \qquad \forall\ v \in V_h.$$

Now, the problem (2.2) can be rewritten in a matrix form

$$A\mathbf{u} = \mathbf{F} \tag{2.6}$$

where A is symmetric and positive definite.

3. Matrix Formulation and Its Properties

Our goal is to introduce an algebraic formulation of the approximate problem using a type of static condensation that eliminates some of the unknowns. In this way we can reduce substantially the size of the problem. For this approach we need a special partitioning of the domain into tetrahedra that have some regularity and preserve the simplicity of the algebraic problem.

First, we partition Ω into cubes with size of the edges $h = 1/n$ and denote them by $C = C^{(i,j,k)}$ where (x_{1i}, x_{2j}, x_{3k}) is the right back upper corner of the cube. This partitioning is denoted by $\mathcal{T}_C$.

Next, we divide each cube $C = C^{(i,j,k)}$ into two prisms $P_1 = P_1^{(i,j,k)}$ and $P_2 = P_2^{(i,j,k)}$ as shown in Fig. 1 and denote this partitioning of Ω by $\mathcal{T}_P$.

Finally, we divide each prism into three tetrahedra as shown in Fig. 1 and denote this partitioning of Ω into tetrahedra by $\mathcal{T}_T$.

Let $P = P^{(i,j,k)} \in \mathcal{T}_P$ be a particular prism of the partition $\mathcal{T}_P$. Denote by V_h^P the subspace of restrictions of the functions in V_h onto P. These restrictions define vectors $\mathbf{u}_P$ that are restrictions of a vector $\mathbf{u} \in \mathbb{R}^N$. The dimension of V_h^P we denote by N^P. Obviously, for prisms with no faces on $\partial\Omega$ the dimension $N^P = 10$.

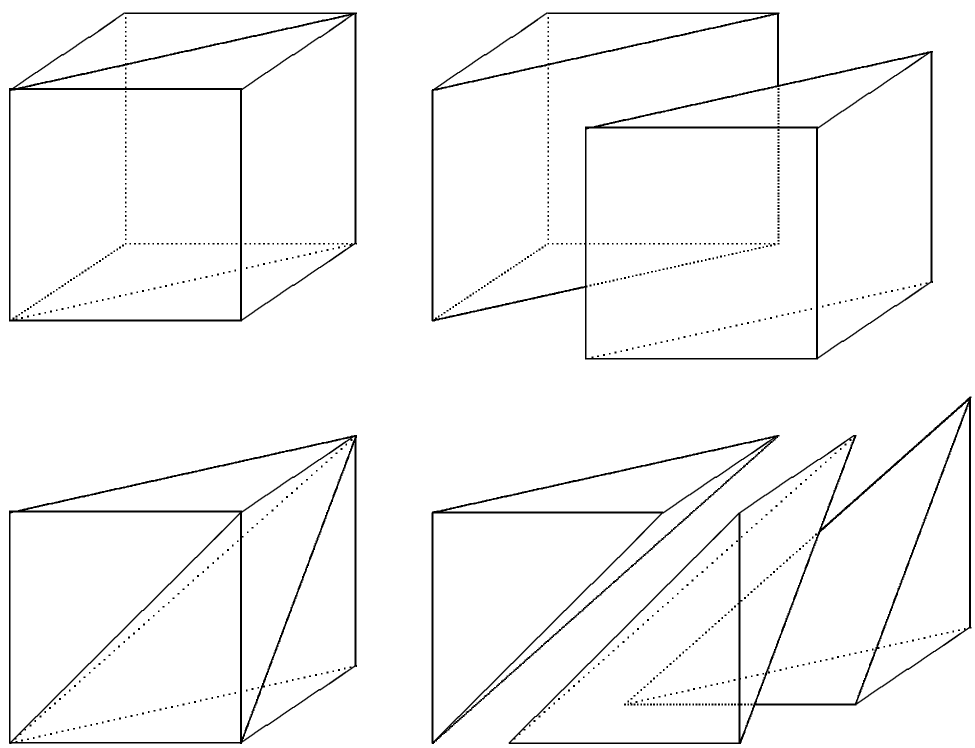


FIGURE 1 *Partition of cube into prisms and tetrahedra.*

Local stiffness matrices A^P on prisms $P \in \mathcal{T}_P$ are defined by

$$(A^P \mathbf{u}_P, \mathbf{v}_P)_N = \sum_{T \subset P} \int_T a(x)\, \nabla u_h \cdot \nabla v_h \; dx \tag{3.1}$$

for any $P \in \mathcal{T}_P$.

Then the global stiffness matrix is determined by assembling the local stiffness matrices. The following equality holds true for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$:

$$(A\mathbf{u}, \mathbf{v})_N = \sum_{P \in \mathcal{T}_P} (A^P \mathbf{u}_P, \mathbf{u}_P)_N. \tag{3.2}$$

Now we consider a prism P of an arbitrary cube that has no face on the boundary $\partial\Omega$ and enumerate the faces s_j, $j = 1, \ldots, 10$ of the tetrahedra in this prism as shown in Fig. 2.

Then the local stiffness matrix of this prism for the case $a(X) \equiv 1$ has the following form:

$$A^P = \frac{3h}{2} \left[\begin{array}{rrrrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 4 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 4 \end{array} \right] \equiv \frac{3h}{2} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.3}$$

where

$$A_{22} = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

Along with matrix A^P we introduce the following matrix B^P defined on the same space V_h^P:

$$B^P = \frac{3h}{2} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & B_{22} \end{bmatrix}, \qquad B_{22} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}. \tag{3.4}$$

It is easy to see that the following holds true.

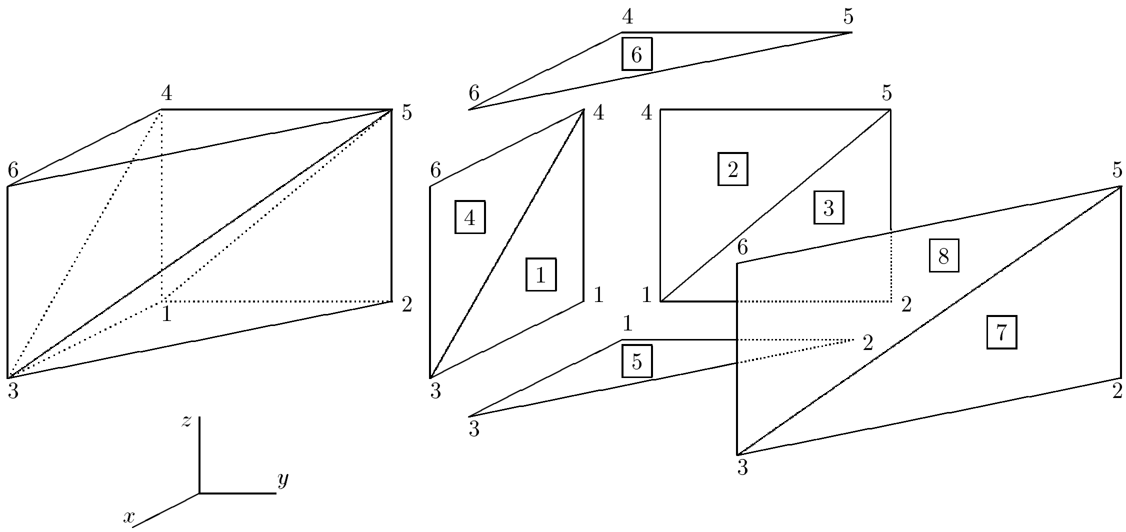
PROPOSITION 1. $ker A^P = ker B^P$.

Remark. If the prism $P \in \mathcal{T}_P$ has a face on $\partial\Omega$, the dimension of the matrix A^P will be less than 10 and the modification of B_{22} is obvious.

Then we define the $N \times N$ matrix B by the following equality:

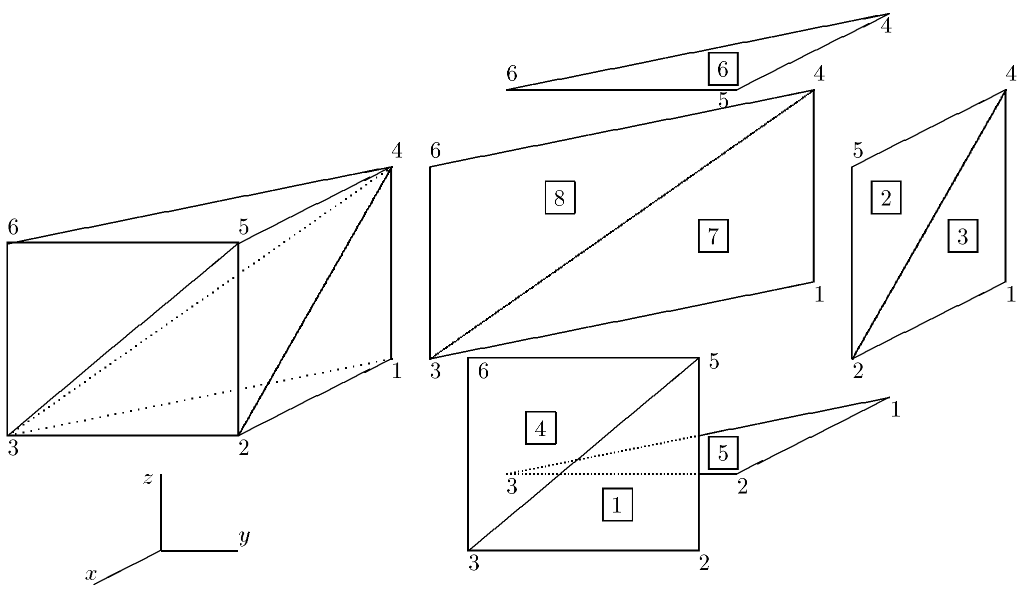
$$(B\mathbf{u}, \mathbf{v})_N = \sum_{P \in \mathcal{T}_P} (B^P \mathbf{u}_P, \mathbf{v}_P)_N \qquad \forall\ \mathbf{u}, \mathbf{v} \in \mathbb{R}^N. \tag{3.5}$$

Since matrix B will be used for preconditioning the original problem (2.6) it is important to estimate the condition number of $B^{-1}A$.



(a) P_1

$s_1 = (1,4,3)$ $s_3 = (1,2,5)$ $s_5 = (1,2,3)$ $s_7 = (2,5,3)$ $s_9 = (1,5,3)$
$s_2 = (1,4,5)$ $s_4 = (3,4,6)$ $s_6 = (4,5,6)$ $s_8 = (3,5,6)$ $s_{10} = (3,4,5)$



(b) P_2

$s_1 = (2,3,5)$ $s_3 = (1,2,4)$ $s_5 = (1,2,3)$ $s_7 = (1,3,4)$ $s_9 = (2,3,4)$
$s_2 = (2,4,5)$ $s_4 = (3,5,6)$ $s_6 = (4,5,6)$ $s_8 = (3,4,6)$ $s_{10} = (3,4,5)$

FIGURE 2. *Local enumeration of faces in a prism.*

Using the fact that all element stiffness matrices are nonnegative and following [15], we easily get the estimates:

$$\max_{u\in\mathbb{R}^N} \frac{(A\mathbf{u},\mathbf{u})}{(B\mathbf{u},\mathbf{u})} \leq \max_{\substack{P\in\mathcal{T}_P \\ (B^P\mathbf{u}_P,\mathbf{u}_P)\neq 0}} \frac{(A^P\mathbf{u}_P,\mathbf{u}_P)}{(B^P\mathbf{u}_P,\mathbf{u}_P)}, \tag{3.6}$$

$$\min_{u\in\mathbb{R}^N} \frac{(A\mathbf{u},\mathbf{u})}{(B\mathbf{u},\mathbf{u})} \geq \min_{\substack{P\in\mathcal{T}_P \\ (B^P\mathbf{u}_P,\mathbf{u}_P)\neq 0}} \frac{(A^P\mathbf{u}_P,\mathbf{u}_P)}{(B^P\mathbf{u}_P,\mathbf{u}_P)}. \tag{3.7}$$

In this way the estimates of the maximal eigenvalue $\lambda_{\max}$ and the minimal eigenvalue $\lambda_{\min}$ of the eigenvalue problem

$$A\mathbf{u} = \lambda B\mathbf{u} \tag{3.8}$$

are estimated from above and below by local analysis, solving the local problems

$$A^P\mathbf{u}_P = \mu B^P\mathbf{u}_P, \qquad (B^P\mathbf{u}_P,\mathbf{u}_P)\neq 0, \qquad P\in\mathcal{T}_P. \tag{3.9}$$

Using superelement analysis it is easy to show that to get the minimal $\mu_{\min}$ and the maximal $\mu_{\max}$ eigenvalues of (3.9) one has to consider the worst case when the prism $P\in\mathcal{T}_P$ has no face on the boundary $\partial\Omega$, i.e., $P\cap\partial\Omega=\emptyset$.

Then a direct calculation verifies the following result.

Proposition 2. Eigenvalues of problem (3.9) lie in the inverval $[2-\sqrt{3}, 2+\sqrt{3}]$.

Then the inequalities (3.6) and (3.7) yield:

Proposition 3. Eigenvalues of problem (3.8) lie in the interval $[2-\sqrt{3}, 2+\sqrt{3}]$ and therefore

$$\mathrm{cond}(B^{-1}A) \leq (2+\sqrt{3})^2.$$

We stress that the condition number of the matrix $B^{-1}A$ is bounded by a constant independent of the step size of the mesh h.

Now we divide all unknowns in the system into two groups:

1. The first group consists of all unknowns corresponding to faces of the prisms in the partition $\mathcal{T}_P$, excluding, of course, the faces on $\partial\Omega$ (see Fig. 2).

2. The second group consists of the unknowns corresponding to the faces of the tetrahedra that are internal for each prism (these are faces s_9 and s_{10} on Fig. 2).

This splitting of the space $\mathbb{R}^N$ induces the following presentation of the vectors $\mathbf{v}^T=(\mathbf{v}_1^T,\mathbf{v}_2^T)$, where $\mathbf{v}_1\in\mathbb{R}^{N_1}$ and $\mathbf{v}_2\in\mathbb{R}^{N_2}$. Obviously, $N_1 = N - 4n^3$. Then matrix B can be presented in the following block form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \qquad \mathrm{dim}B_{11} = N_1. \tag{3.10}$$

Denote now by $\hat{B}_{11} = B_{11} - B_{12}B_{22}^{-1}B_{21}$ the Schur complement of B obtained by elimination of the vector $\mathbf{v}_2$. Then $B_{11} = \hat{B}_{11} + B_{12}B_{22}^{-1}B_{21}$, so the matrix B has the form

$$B = \begin{bmatrix} \hat{B}_{11} + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \tag{3.11}$$

Note that for each prism $P \in \mathcal{T}_P$ the unknowns on the faces s_9 and s_{10} (see Fig. 2) are connected through the equation $B\mathbf{v} = F$, only with the unknowns associated with this prism and therefore can be eliminated locally; i.e., the matrix B_{22} is block diagonal with 2×2 blocks and can be inverted locally (prism by prism). Thus matrix $\hat{B}_{11}$ is easily computable.

4. Multilevel Substructuring Preconditioner

In this section we will propose two modifications of matrix B (3.11) in the form

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

and consider their properties and computational schemes.

4.1. Group Partitioning of the Grid Points

For the sake of simplicity of representation of matrices and computational schemes we introduce the following partitioning of all nodes of $\overset{\circ}{Q}_h$ into three groups.

Let us denote by $s_{r,l,m}^{(i,j,k)}$ the face of the cube $C^{(i,j,k)}$ with vertices r, l, m (see Fig. 3) and partition the nodes of $\overset{\circ}{Q}_h$ in the following manner:

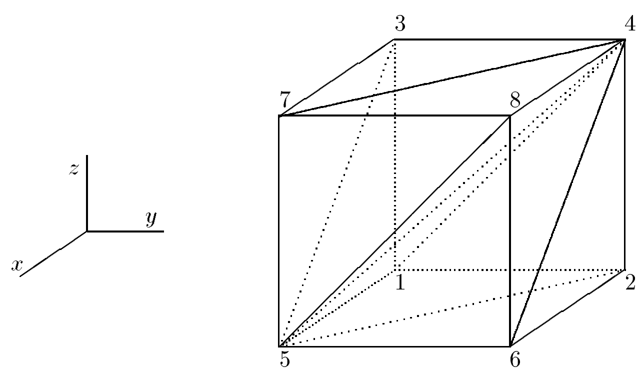


FIGURE 3. *Cube* $c^{(i,j,k)}$.

1. First, we group the nodes on the faces

$$s_{2,4,5}^{(i,j,k)} \quad \text{and} \quad s_{4,5,7}^{(i,j,k)}, \qquad i,j,k = \overline{1,n}.$$

We denote the unknowns at these nodes by $VI_\ell^{(i,j,k)}$, $\ell = 1,2$, $i,j,k = \overline{1,n}$.

2. Then, we take the nodes on the faces perpendicular to the x, y, and z axes:

$$\text{(i)} \quad s_{1,2,4}^{(i,j,k)}, \quad s_{1,3,4}^{(i,j,k)}, \quad i = \overline{2,n}, \quad j,k = \overline{1,n}.$$

We denote the unknowns at these nodes by $Vx_\ell^{(i,j,k)}$, $\ell = 1,2$, $i = \overline{2,n}$, $j,k = \overline{1,n}$;

$$\text{(ii)} \quad s_{1,3,5}^{(i,j,k)}, \quad s_{5,3,7}^{(i,j,k)}, \quad j = \overline{2,n}, \quad i,k = \overline{1,n}. \tag{4.1}$$

We denote the unknowns at these nodes by $Vy_\ell^{(i,j,k)}$, $\ell = 1,2$, $j = \overline{2,n}$, $i,k = \overline{1,n}$;

$$\text{(iii)} \quad s_{1,2,5}^{(i,j,k)}, \quad s_{2,5,6}^{(i,j,k)}, \quad i,j = \overline{1,n}, \quad k = \overline{2,n}.$$

We denote the unknowns at these nodes by $Vz_\ell^{(i,j,k)}$, $\ell = 1,2$, $i,j = \overline{1,n}$, $k = \overline{2,n}$.

3. At last, we take the remaining nodes on faces

$$s_{1,4,5}^{(1,j,k)}, \quad s_{3,4,5}^{(i,j,k)}, \quad s_{4,5,6}^{(i,j,k)}, \quad s_{4,5,8}^{(i,j,k)}, \quad i,j,k = \overline{1,n}.$$

We denote the unknowns at these nodes by $VA_\ell^{(i,j,k)}$, $\ell = \overline{1,n}$, $i,j,k = \overline{1,n}$.

4.2. Three Level Preconditioner: Variant I

Let us take an arbitrary cube $C^{(i,j,k)}$ that is partitioned into left and right prisms (see Fig. 1) $P_p^{(i,j,k)}$, $p = 1,2$. Below we skip indices '(i,j,k)' and 'p' for simplicity of notation in all cases where there is no ambiguity.

In the local numeration (Fig. 2) matrices B_1 and B_2 correspond to prisms having the form (3.3)-(3.4). We rewrite these matrices in the ordering (4.1) introduced above in this section:

$$
\begin{aligned}
B_1 &= \left(\frac{3h}{2}\right)\left[\begin{array}{rr|rrrrrr|rr}
3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
\hline
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
\hline
-1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 3 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 3
\end{array}\right], \\
B_2 &= \left(\frac{3h}{2}\right)\left[\begin{array}{rr|rrrrrr|rr}
3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & 0 \\
-1 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
\hline
-1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 3 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 3
\end{array}\right].
\end{aligned} \tag{4.2}
$$

The partitioning of nodes into groups (4.1) induces block forms of matrices B_p, $p = 1, 2$:

$$
B_p = \left[\begin{array}{cc} B_{11,p} & B_{12,p} \\ B_{21,p} & B_{22,p} \end{array}\right], \qquad p = 1, 2, \tag{4.3}
$$

where blocks $B_{22,p}$ correspond to the unknowns of the 3-d group and blocks $B_{11,p}$ correspond to the unknowns of the first and second groups.

We eliminate the unknowns of the 3-d group from each matrix B_p, $p = 1, 2$ which is done locally on each prism. Then we get the matrices

$$
\hat{B}_{11,p} = B_{11,p} - B_{12,p} B_{22,p}^{-1} B_{21,p}, \qquad p = 1, 2,
$$

where $p = 1$ corresponds to a right prism and $p = 2$ to a left prism

$$
\hat{B}_{11,1} = \left(\frac{3h}{2}\right) \left[\begin{array}{rr|rrrrrr}
8/3 & -1 & -1 & 0 & -1/3 & 0 & -1/3 & 0 \\
-1 & 8/3 & 0 & -1/3 & 0 & -1 & 0 & -1/3 \\
\hline
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/3 & 0 & 2/3 & 0 & 0 & 0 & -1/3 \\
-1/3 & 0 & 0 & 0 & 2/3 & 0 & -1/3 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1/3 & 0 & 0 & 0 & -1/3 & 0 & 2/3 & 0 \\
0 & -1/3 & 0 & -1/3 & 0 & 0 & 0 & 2/3
\end{array}\right],
$$

$$
\hat{B}_{11,2} = \left(\frac{3h}{2}\right) \left[\begin{array}{rr|rrrrrr}
8/3 & -1 & -1/3 & 0 & -1 & 0 & -1/3 & 0 \\
-1 & 8/3 & 0 & -1 & 0 & -1/3 & 0 & -1/3 \\
\hline
-1/3 & 0 & 2/3 & 0 & 0 & 0 & -1/3 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1/3 & 0 & 0 & 0 & 2/3 & 0 & -1/3 \\
-1/3 & 0 & -1/3 & 0 & 0 & 0 & 2/3 & 0 \\
0 & -1/3 & 0 & 0 & 0 & -1/3 & 0 & 2/3
\end{array}\right]. \tag{4.4}
$$

Together with $\hat{B}_{11,p}$, $p = 1, 2$, we define on each cube matrices $B_{1,p}$, $p = 1, 2$, in the following way:

$$
B_{1,1} = \left(\frac{3h}{2}\right) \left[\begin{array}{rr|rrrrrr}
8/3 & -1 & -1 & 0 & -1/3 & 0 & -1/3 & 0 \\
-1 & 8/3 & 0 & -1/3 & 0 & -1 & 0 & -1/3 \\
\hline
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\
-1/3 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1/3 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \\
0 & -1/3 & 0 & 0 & 0 & 0 & 0 & 1/3
\end{array}\right],
$$

$$
B_{1,2} = \left(\frac{3h}{2}\right) \left[\begin{array}{rr|rrrrrr}
8/3 & -1 & -1/3 & 0 & -1 & 0 & -1/3 & 0 \\
-1 & 8/3 & 0 & -1 & 0 & -1/3 & 0 & -1/3 \\
\hline
-1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1/3 & 0 & 0 & 0 & 1/3 & 0 & 0 \\
-1/3 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \\
0 & -1/3 & 0 & 0 & 0 & 0 & 0 & 1/3
\end{array}\right]. \tag{4.5}
$$

Both matrices $\hat{B}_{11,p}$ and $B_{1,p}$, $p = 1, 2$, are irreducible and have the same kernel, i.e., $\ker \hat{B}_{11,p} = \ker B_{1,p}$. It is easy to show that eigenvalues of the spectral problems (for both left and right prisms)

$$
\hat{B}_{11,p}\mathbf{u} = \mu B_{1,p}\mathbf{u}, \quad \mathbf{u} \in \mathbb{R}^8, \qquad (B_{1,p}\mathbf{u}, \mathbf{u}) \neq 0, \quad p = 1, 2, \tag{4.6}
$$

belong to the interval $\mu \in [1, 3]$.

Now we define a new matrix on each prism:

$$\tilde{B}_p = \begin{bmatrix} B_{1,p} + B_{12,p}B_{22,p}^{-1}B_{21,p} & B_{12,p} \\ B_{21,p} & B_{22,p} \end{bmatrix}, \qquad p = 1, 2. \tag{4.7}$$

In the case when cube C has nonempty intersection with $\partial\Omega$ all considerations are the same; matrices $B_{1,p}$, $B_{12,p}$, $B_{21,p}$, $p = 1, 2$ will not have rows and columns corresponding to the nodes on the boundary.

Define eigenvalue problems for each prism P:

$$B_P\mathbf{u} = \xi\tilde{B}_P\mathbf{u}, \qquad (\tilde{B}_P\mathbf{u}, \mathbf{u}) \neq 0, \qquad P \in \mathcal{T}_P. \tag{4.8}$$

Remark. For inner prisms, that is for prisms which have no face on $\partial\Omega$, $\mathbf{u} \in \mathbb{R}^{10}$. Because the eigenvalues of the problems (4.6) belong to the interval $[1, 3]$, the same is true for the problems (4.8); thus we can formlate the following:

Proposition 4. Eigenvalues of the problems (4.8) belong to the interval $[1, 3]$. Moreover, the eigenvalue problems on each prism

$$A_P\mathbf{u} = \nu\tilde{B}_P\mathbf{u}, \qquad (\tilde{B}_P\mathbf{u}, \mathbf{u}) \neq 0, \qquad P \in \mathcal{T}_P, \tag{4.9}$$

has eigenvalues ν that belong to the interval

$$\nu \in [2 - \sqrt{3}, 3(2 + \sqrt{3})].$$

Now we define the symmetric positive-definite $N_1 \times N_1$ matrix $\tilde{B}_1$ by

$$(\tilde{B}_1\mathbf{u}_1, \mathbf{v}_1) = \sum_{P \in \mathcal{T}_P} (\tilde{B}_P\mathbf{u}_{1,P}, \mathbf{v}_{1,P}), \tag{4.10}$$

where the vectors $\mathbf{v}_1, \mathbf{u}_1 \in \mathbb{R}^{N_1}$, and $\mathbf{u}_{1,P}, \mathbf{v}_{1,P}$ are the restrictions of the vectors $\mathbf{u}_1, \mathbf{v}_1$ on the prism P.

Along with matrix B in the form of (3.11), we introduce the matrix

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \tag{4.11}$$

Again, using superelement analysis and Proposition 4, it is easy to prove the following theorem.

Theorem 1. *Matrix $\tilde{B}$, defined in (4.11), is spectrally equivalent to matrix A; i.e.,*

$$\alpha\tilde{B} \leq A \leq \beta\tilde{B},$$

with $\alpha = (2 - \sqrt{3})$ and $\beta = 3(2 + \sqrt{3})$. Therefore,

$$\mathrm{cond}(\tilde{B}^{-1}A) \leq 3(2 + \sqrt{3})^2. \tag{4.12}$$

Instead of matrix B in the form of (3.11) we will take the matrix $\tilde{B}$ in the form of (4.11) as the two-level preconditioner for matrix A. As we noted earlier, matrix B_{22} is block-diagonal and can be inverted locally on prisms.

Now consider the linear system

$$\tilde{B}_1 \mathbf{u} = \mathbf{f}. \tag{4.13}$$

In terms of the partitioning (4.1) of nodes, the matrix $\tilde{B}_1$ has the block form

$$\tilde{B}_1 = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \tag{4.14}$$

where the block C_{22} corresponds to the nodes from group (2), which are on the faces of tetrahedra perpendicular to the coorindate axis. It can be shown that matrix C_{22} is diagonal. In partitioning (4.1), we present $\mathbf{u}$ and $\mathbf{f}$ of (4.13) in the form

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}. \tag{4.15}$$

Then, after elimination of the second group of unknowns,

$$\mathbf{u}_2 = C_{22}^{-1}(\mathbf{f}_2 - C_{21}\mathbf{u}_1)$$

we get the system of linear equations

$$(C_{11} - C_{12}C_{22}^{-1}C_{21})\mathbf{u}_1 = \mathbf{f}_1 - C_{12}C_{22}^{-1}\mathbf{f}_2 = \tilde{\mathbf{f}}_1, \tag{4.16}$$

where the vector $\mathbf{u}_1$ and the block C_{11} correspond to the unknowns from the first group, which have only two unknowns per each cube. The dimension of vectors $\mathbf{u}_1$ and $\mathbf{f}_1$ is equal to

$$\dim(\mathbf{u}_1) = 2n^3. \tag{4.17}$$

Because we have introduced a two-level subdivision of matrix $\tilde{B}_1$, original matrix $\tilde{B}$ can be considered as a three-level preconditioner.

4.3. Computational Scheme: Variant I

Let us write explicity the equations from (4.13) in terms of the unknowns introduced in (4.1), i.e., in the terms of

$$fI_\ell^{(i,j,k)}, \quad UI_\ell^{(i,j,k)}, \quad \ell = 1,2, \quad i,j,k = \overline{1,n};$$

$$fx_\ell^{(i,j,k)}, \quad Ux_\ell^{(i,j,k)}, \quad \ell = 1,2, \quad i = \overline{2,n}, \qquad j,k = \overline{1,n};$$

$$fy_\ell^{(i,j,k)}, \quad Uy_\ell^{(i,j,k)}, \quad \ell = 1,2, \quad j = \overline{2,n}, \qquad i,k = \overline{1,n};$$

$$fz_\ell^{(i,j,k)}, \quad Uz_\ell^{(i,j,k)}, \quad \ell = 1,2, \quad k = \overline{2,n}, \qquad i,j = \overline{1,n}.$$

To simplify the representation of these equations we write (4.13) for the case of $a(x) \equiv 1$.

$$
\begin{aligned}
&\begin{bmatrix} 16/3 & -2 \\ -2 & 16/3 \end{bmatrix} \mathbf{UI}^{(i,j,k)} - \\
&\quad -(1-\delta_{in}) \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{Ux}^{(i,j,k)} - (1-\delta_{i1}) \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \mathbf{Ux}^{(i-1,j,k)} \\
&\quad -(1-\delta_{jn}) \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \mathbf{Uy}^{(i,j,k)} - (1-\delta_{j1}) \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{Uy}^{(i,j-1,k)} \\
&-(1-\delta_{kn})\frac{1}{3}\mathbf{Uz}^{(i,j,k)} - (1-\delta_{k1})\frac{1}{3}\mathbf{Uz}^{(i,j,k-1)} = \left(\frac{2}{3h}\right) \mathbf{fI}^{(i,j,k)}, \qquad i,j,k = \overline{1,n},
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
\frac{4}{3}\mathbf{Ux}^{(i,j,k)} - \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \mathbf{UI}^{(i-1,j,k)} - \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{UI}^{(i,j,k)} &= \left(\frac{2}{3h}\right) \mathbf{fx}^{(i,j,k)}, \\
& i = \overline{2,n}, \quad j,k = \overline{1,n}, \\
\frac{4}{3}\mathbf{Uy}^{(i,j,k)} - \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{UI}^{(i,j-1,k)} - \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \mathbf{UI}^{(i,j,k)} &= \left(\frac{2}{3h}\right) \mathbf{fy}^{(i,j,k)}, \\
& j = \overline{2,n}, \quad i,k = \overline{1,n}, \\
\frac{2}{3}\mathbf{Uz}^{(i,j,k)} - \frac{1}{3}\mathbf{UI}^{(i,j,k-1)} - \frac{1}{3}\mathbf{UI}^{(i,j,k)} &= \left(\frac{2}{3h}\right) \mathbf{fz}^{(i,j,k)}, \\
& k = \overline{2,n}, \quad i,j = \overline{1,n},
\end{aligned} \tag{4.19}
$$

where the function $\delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$ is introduced to take into account the Dirichlet boundary conditions, and any vector $\mathbf{vr}^{(i,j,k)} = \begin{bmatrix} vr_1^{(i,j,k)} \\ vr_2^{(i,j,k)} \end{bmatrix} \in \mathbb{R}^2$.

After eliminating the unknowns $Ux_\ell^{(i,j,k)}, Uy_\ell^{(i,j,k)}, Uz_\ell^{(i,j,k)}$ from equations (4.18), we will have a block "seven-point" computational scheme with the 2×2-blocks for the unknowns $UI_\ell^{(i,j,k)}$.

From (4.19) we have

$$\begin{aligned}
\mathbf{Ux}^{(i,j,k)} &= \frac{3}{4}\left(\frac{2}{3h}\right)\mathbf{fx}^{(i,j,k)} + \frac{3}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}\mathbf{UI}^{(i-1,j,k)} + \frac{3}{4}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{UI}^{(i,j,k)}, \\
&\quad i = \overline{2,n}, \quad j,k = \overline{1,n}, \\
\mathbf{Uy}^{(i,j,k)} &= \frac{3}{4}\left(\frac{2}{3h}\right)\mathbf{fy}^{(i,j,k)} + \frac{3}{4}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{UI}^{(i,j-1,k)} + \frac{3}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}\mathbf{UI}^{(i,j,k)}, \\
&\quad i,k = \overline{1,n}, \quad j = \overline{2,n}, \\
\mathbf{Uz}^{(i,j,k)} &= \frac{3}{2}\left(\frac{2}{3h}\right)\mathbf{fz}^{(i,j,k)} + \frac{1}{2}\mathbf{UI}^{(i,j,k-1)} + \frac{1}{2}\mathbf{UI}^{(i,j,k)}, \\
&\quad i,j = \overline{1,n}, \quad k = \overline{2,n}.
\end{aligned} \tag{4.20}$$

Substituting these expressions for $\mathbf{Ux}^{(i,j,k)}$, $\mathbf{Uy}^{(i,j,k)}$, and $\mathbf{Uz}^{(i,j,k)}$ into (4.18) we get the equations

$$\begin{aligned}
&\begin{bmatrix} 16/3 & -2 \\ -2 & 16/3 \end{bmatrix}\mathbf{UI}^{(i,j,k)} \\
&-(1-\delta_{i1})\left(\frac{1}{4}\mathbf{UI}^{(i-1,j,k)} + \begin{bmatrix} 3/4 & 0 \\ 0 & 1/12 \end{bmatrix}\mathbf{UI}^{(i,j,k)}\right) \\
&-(1-\delta_{in})\left(\frac{1}{4}\mathbf{UI}^{(1+1,j,k)} + \begin{bmatrix} 1/12 & 0 \\ 0 & 3/4 \end{bmatrix}\mathbf{UI}^{(i,j,k)}\right) \\
&-(1-\delta_{j1})\left(\frac{1}{4}\mathbf{UI}^{(i,j-1,k)} + \begin{bmatrix} 1/12 & 0 \\ 0 & 3/4 \end{bmatrix}\mathbf{UI}^{(i,j,k)}\right) \\
&-(1-\delta_{jn})\left(\frac{1}{4}\mathbf{UI}^{(i,j+1,k)} + \begin{bmatrix} 3/4 & 0 \\ 0 & 1/12 \end{bmatrix}\mathbf{UI}^{(i,j,k)}\right) \\
&-(1-\delta_{k1})\frac{1}{6}\left(\mathbf{UI}^{(i,j,k-1)} + \mathbf{UI}^{(i,j,k)}\right) - (1-\delta_{kn})\frac{1}{6}\left(\mathbf{UI}^{(i,j,k+1)} + \mathbf{UI}^{(i,j,k)}\right) \\
&= \mathbf{F}^{(i,j,k)}, \qquad ij,k = \overline{1,n},
\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}
\mathbf{F}^{(i,j,k)} &= \left(\frac{2}{3h}\right)\Big\{\mathbf{fI}^{(i,j,k)} + \\
&+(1-\delta_{i1})\frac{3}{4}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{fx}^{(i-1,j,k)} + (1-\delta_{in})\frac{3}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}\mathbf{fx}^{(i+1,j,k)} + \\
&+(1-\delta_{j1})\frac{3}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}\mathbf{fy}^{(i,j-1,k)} + (1+\delta_{jn})\frac{3}{4}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{fy}^{(i,j+1,k)} + \\
&+(1-\delta_{k1})\frac{1}{2}\mathbf{fz}^{(1,j,k-1)} + (1-\delta_{kn})\frac{1}{2}\mathbf{fz}^{(i,j,k+1)}\Big\}, \qquad i,j,k = \overline{1,n}.
\end{aligned} \tag{4.22}$$

Thus, for solving the linear system with matrix $\tilde{B}_1$ (4.13), we first solve problem (4.21) for $\mathbf{u}_1$ with $(2n \times 2n)$-seven-block-diagonal matrix and after that compute the vector $\mathbf{u}_2$ from (4.20).

Unfortunately, the matrix of linear system (4.21) has a rather complicated form which makes the solution rather difficult. Below we show that if on each prism instead of matrices (4.5) we introduce another matrix $B_1^{(i,j,k)}$, then we will have as a result a simlier matrix (4.21). In that case for solving the sytem for unknowns $\mathbf{u}_1$, we can use the method of separation of variables or another fast method.

4.4. Three-Level Preconditioner: Variant II

Instead of matrices (4.5) we introduce other matrices $B_{1,p} = \hat{B}_1$, $p = 1, 2$:

$$\hat{B}_1 = \left(\frac{3h}{2}\right) \left[\begin{array}{rr|rrrrrr} 8/3 & -1 & -2/3 & 0 & -2/3 & 0 & -1/6 & -1/6 \\ -1 & 8/3 & 0 & -2/3 & 0 & -2/3 & -1/6 & -1/6 \\ \hline -2/3 & 0 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 2/3 & 0 & 0 & 0 & 0 \\ -2/3 & 0 & 0 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 0 & & 2/3 & 0 & 0 \\ -1/6 & -1/6 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ -1/6 & -1/6 & 0 & 0 & 0 & 0 & 0 & 1/3 \end{array} \right] . \tag{4.23}$$

This matrix is irreducible and has the same kernel as matrices $B_{11,p}$, $p = 1, 2$; that is

$$\ker B_{11,p} = \ker \hat{B}_1, \qquad p = 1, 2.$$

Then we have the following

Proposition 5. For any $P \in \mathcal{T}_P$ the eigenvalues of the spectral problems

$$B_{11,P}\mathbf{u} = \mu \hat{B}_1 \mathbf{u}, \qquad (\hat{B}_1, \mathbf{u}, \mathbf{u}) \neq 0,$$

belong to the interval

$$\mu \in \left[\frac{5}{11}(3 - \sqrt{3}), \frac{3}{5}(3 + \sqrt{3}) \right] . \tag{4.24}$$

Again, we define the matrices $\hat{B}_P$ (similar to (4.7))

$$\hat{B}_P = \begin{bmatrix} \hat{B}_1 + B_{12,P} B_{22,P}^{-1} B_{21,P} & B_{12,P} \\ B_{21,P} & B_{22,P} \end{bmatrix}, \qquad P \in \mathcal{T}_P \tag{4.25}$$

and consider the eigenvalue problems (4.8). For those problems we can formulate the following statement.

Proposition 6. Eigenvalues of the problems (4.8) with the new block $\hat{B}_1$ belong to the interval defined in (4.24). For the same reason, eigenvalues of the spectral problems (4.9) for each prism belong to the interval

$$\nu \in \left[\frac{5}{11}(3 - \sqrt{3})(2 - \sqrt{3}), \frac{3}{5}(3 + \sqrt{3})(2 + \sqrt{3}) \right] .$$

Now we define another symmetric positive-definite $N_1 \times N_1$-matrix $\hat{B}_1$, using equality (4.10) with $\hat{B}_P$ instead of $\tilde{B}_P$. Then the new preconditioner $\hat{B}$ is defined by

$$\hat{B} = \begin{bmatrix} \hat{B}_1 + B_{12} B_{22}^{-1} B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \tag{4.26}$$

In the same way as in Section 4.2, using superelement analysis and Proposition 6, we prove the following:

Theorem 2. *Matrix $\hat{B}$ defined in (4.26) with the new blocks $\hat{B}_1$ in the form of (4.23) is spectrally equivalent to matrix A; i.e.,*

$$\alpha \hat{B} \leq A \leq \beta \hat{B}$$

with $\alpha = \frac{5}{11}(3-\sqrt{3})(2-\sqrt{3})$ and $\beta = \frac{3}{5}(3+\sqrt{3})(2+\sqrt{3})$. Therefore,

$$\operatorname{cond}(\hat{B}^{-1} A) \leq \nu, \tag{4.27}$$

where $\nu = 5(2+\sqrt{3})^2$.

Let us now consider the linear system with matrix $\hat{B}_1$:

$$\hat{B}_1 \mathbf{u} = \mathbf{f}. \tag{4.28}$$

Similar to matrix $\tilde{B}_1$, matrix $\hat{B}_1$ can be represented in the block form

$$\hat{B}_1 = \begin{bmatrix} C_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix}, \tag{4.29}$$

where block C_{11} coincides with the same block of matrix $\tilde{B}_1$ (4.14), and matrix $\hat{C}_{22}$ is diagonal.

4.5. Computational Scheme: Variant II

Again, we write equations (4.28) explicitly in terms of the unknowns introduced in (4.1) for the case $a(x) \equiv 1$:

$$\begin{aligned} &\begin{bmatrix} 16/3 & -2 \\ -2 & 16/3 \end{bmatrix} \mathbf{UI}^{(i,j,k)} - (1-\delta_{i1})\frac{2}{3}\mathbf{Ux}^{(i-1,j,k)} - (1-\delta_{in})\frac{2}{3}\mathbf{Ux}^{(i,j,k)} \\ &\quad -(1-\delta_{j1})\frac{2}{3}\mathbf{Uy}^{(i,j-1,k)} - (1-\delta_{jn})\frac{2}{3}\mathbf{Uy}^{(i,j,k)} \\ &\quad -(1-\delta_{k1})\frac{1}{6}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{Uz}^{(i,j,k-1)} - (1-\delta_{kn})\frac{1}{6}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{Uz}^{(i,j,k)} \\ &\qquad = \left(\frac{2}{3h}\right) \mathbf{fI}^{(i,j,k)}, \qquad i,j,k = \overline{1,n}, \end{aligned} \tag{4.30}$$

$$\frac{4}{3}\mathbf{Ux}^{(i,j,k)} - \tfrac{2}{3}\mathbf{UI}^{(i-1,j,k)} - \tfrac{2}{3}\mathbf{UI}^{(i,j,k)} = \left(\frac{2}{3h}\right)\mathbf{fx}^{(i,j,k)}, \quad i=\overline{2,n}, \quad j,k=\overline{1,n},$$
$$\frac{4}{3}\mathbf{Uy}^{(i,j,k)} - \tfrac{2}{3}\mathbf{UI}^{(i,j-1,k)} - \tfrac{2}{3}\mathbf{UI}^{(i,j,k)} = \left(\frac{2}{3h}\right)\mathbf{fy}^{(i,j,k)}, \quad j=\overline{2,n}, \quad i,k=\overline{1,n},$$
$$\frac{2}{3}\mathbf{Uz}^{(i,j,k)} - \tfrac{1}{6}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{UI}^{(i,j,k-1)} - \tfrac{1}{6}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{UI}^{(i,j,k)} = \tfrac{2}{3h}\mathbf{fz}^{(i,j,k)},$$
$$k=\overline{2,n}, \quad i,j=\overline{1,n}. \tag{4.31}$$

Here the function $\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$ is introduced to take into account the Dirichlet boundary conditions. Eliminating unknowns $Ux_\ell^{(i,j,k)}$, $Uy_\ell^{(i,j,k)}$, $Uz_\ell^{(i,j,k)}$, $\ell=1,2$, from equations (4.30), we will get the block "seven-point" scheme with 2×2-blocks for the unknowns $UI_\ell^{(i,j,k)}$, $\ell=1,2$, $i,j,k=\overline{1,n}$. From (4.31) we have

$$\mathbf{Ux}^{(i,j,k)} = \frac{3}{4}\left(\frac{2}{3h}\right)\mathbf{fx}^{(i,j,k)} + \frac{1}{2}\mathbf{UI}^{(i-1,j,k)} + \frac{1}{2}\mathbf{UI}^{(i,j,k)}, \qquad i=\overline{2,n}, \quad j,k=\overline{1,n},$$
$$\mathbf{Uy}^{(i,j,k)} = \frac{3}{4}\left(\frac{2}{3h}\right)\mathbf{fy}^{(i,j,k)} + \frac{1}{2}\mathbf{UI}^{(i,j-1,k)} + \frac{1}{2}\mathbf{UI}^{(i,j,k)}, \qquad j=\overline{2,n}, \quad i,k=\overline{1,n},$$
$$\mathbf{Uz}^{(i,j,k)} = \frac{3}{2}\left(\frac{2}{3h}\right)\mathbf{fz}^{(i,j,k)} + \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{UI}^{(i,j,k-1)} + \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{UI}^{(i,j,k)},$$
$$k=\overline{2,n}, \quad i,j=\overline{1,n}. \tag{4.32}$$

Substituting (4.32) into (4.30) we get

$$\begin{bmatrix} 16/3 & -2 \\ -2 & 16/3 \end{bmatrix}\mathbf{UI}^{(i,j,k)} -$$
$$-(1-\delta_{i1})\frac{1}{3}\left(\mathbf{UI}^{(i-1,j,k)} + \mathbf{UI}^{(i,j,k)}\right) - (1-\delta_{in})\frac{1}{3}\left(\mathbf{UI}^{(i+1,j,k)} + \mathbf{UI}^{(i,j,k)}\right)$$
$$-(1-\delta_{j1})\frac{1}{3}\left(\mathbf{UI}^{(i,j-1,k)} + \mathbf{UI}^{(i,j,k)}\right) - (1-\delta_{jn})\frac{1}{3}\left(\mathbf{UI}^{(i,j+1,k)} + \mathbf{UI}^{(i,j,k)}\right)$$
$$-(1-\delta_{k1})\frac{1}{12}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left(\mathbf{UI}^{(i,j,k-1)} + \mathbf{UI}^{(i,j,k)}\right)$$
$$-(1-\delta_{kn})\frac{1}{12}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left(\mathbf{UI}^{(i,j,k+1)} + \mathbf{UI}^{(i,j,k)}\right) = \mathbf{F}^{(i,j,k)}, \qquad i,j,k=\overline{1,n}, \tag{4.33}$$

where

$$\mathbf{F}^{(i,j,k)} = \left(\frac{2}{3h}\right)\Big\{\mathbf{fI}^{(i,j,k)} + (1-\delta_{i1})\frac{1}{2}\mathbf{fx}^{(i-1,j,k)} + (1-\delta_{in})\frac{1}{2}\mathbf{fx}^{(i,j,k)}$$
$$+(1-\delta_{j1})\frac{1}{2}\mathbf{fy}^{(i,j-1,k)} + (1-\delta_{jn})\frac{1}{2}\mathbf{fy}^{(i,j,k)}$$
$$+(1-\delta_{k1})\frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{fz}^{(i,j,k-1)} + (1-\delta_{kn})\frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{fz}^{(i,j,k)}\Big\}. \tag{4.34}$$

For solving system (4.33) we introduce the rotation matrix $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and new vectors $\mathbf{V}^{(i,j,k)} = (V_1^{(i,j,k)}\ V_2^{(i,j,k)})^T$, $i,j,k = \overline{1,n}$ such that

$$\mathbf{V}^{(i,j,k)} = Q \cdot \mathbf{UI}^{(i,j,k)}, \qquad i,j,k = \overline{1,n}. \tag{4.35}$$

Then replacing $\mathbf{UI}^{(i,j,k)}$ in (4.33) by

$$\mathbf{UI}^{(i,j,k)} = Q^T \cdot \mathbf{V}^{(i,j,k)}, \qquad i,j,k = \overline{1,n}, \tag{4.36}$$

and multiplying both sides of each matrix equation (4.33) by matrix Q we get the following problem for the unknowns $\mathbf{V}^{(i,j,k)}$:

$$\begin{bmatrix} 10/3 & 0 \\ 0 & 22/3 \end{bmatrix}\mathbf{V}^{(i,j,k)} -$$
$$-(1-\delta_{i1})\frac{1}{3}\left(\mathbf{V}^{(i-1,j,k)} + \mathbf{V}^{(i,j,k)}\right) - (1-\delta_{in})\frac{1}{3}\left(\mathbf{V}^{(i+1,j,k)} + \mathbf{V}^{(i,j,k)}\right)$$
$$-(1-\delta_{j1})\frac{1}{3}\left(\mathbf{V}^{(i,j-1,k)} + \mathbf{V}^{(i,j,k)}\right) - (1-\delta_{jn})\frac{1}{3}\left(\mathbf{V}^{(i,j+1,k)} + \mathbf{V}^{(i,j,k)}\right)$$
$$-(1-\delta_{k1})\frac{1}{6}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\mathbf{V}^{(i,j,k-1)} + \mathbf{V}^{(i,j,k)}\right)$$
$$-(1-\delta_{kn})\frac{1}{6}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\mathbf{V}^{(i,j,k+1)} + \mathbf{V}^{(i,j,k)}\right) = Q \cdot \mathbf{F}^{(i,j,k)} = \tilde{\mathbf{F}}^{(i,j,k)}, \qquad i,j,k = \overline{1,n}. \tag{4.37}$$

It is easy to see that problem (4.37) is decomposed into the following two independent problems:

$$\frac{10}{3}V_1^{(i,j,k)} - (1-\delta_{i1})\frac{1}{3}\left(V_1^{(i-1,j,k)} + V_1^{(i,j,k)}\right) - (1-\delta_{in})\frac{1}{3}\left(V_1^{(i+1,j,k)} + V_1^{(i,j,k)}\right)$$
$$-(1-\delta_{j1})\frac{1}{3}\left(V_1^{(i,j-1,k)} + V_1^{(i,j,k)}\right) - (1-\delta_{jn})\frac{1}{3}\left(V_1^{(i,j+1,k)} + V_1^{(i,j,k)}\right)$$
$$-(1-\delta_{k1})\frac{1}{6}\left(V_1^{(i,j,k-1)} + V_1^{(i,j,k)}\right) - (1-\delta_{kn})\frac{1}{6}\left(V_1^{(i,j,k+1)} + V_1^{(i,j,k)}\right)$$
$$= \tilde{F}_1^{(i,j,k)}, \qquad i,j,k = 1,n \tag{4.38}$$

and

$$\frac{22}{3}V_2^{(i,j,k)} - (1-\delta_{i1})\frac{1}{3}\left(V_2^{(i-1,j,k)} + V_2^{(i,j,k)}\right) - (1-\delta_{in})\frac{1}{3}\left(V_2^{(i+1,j,k)} + V_2^{(i,j,k)}\right)$$
$$-(1-\delta_{j1})\frac{1}{3}\left(V_2^{(i,j-1,k)} + V_2^{(i,j,k)}\right) - (1-\delta_{jn})\frac{1}{3}\left(V_2^{(i,j+1,k)} + V_2^{(i,j,k)}\right)$$
$$= \tilde{F}_2^{(i,j,k)}, \qquad i,j = \overline{1,n}, \quad \forall\ k = \overline{1,n}. \tag{4.39}$$

That is, we reduced the linear system (4.33) of dimension $(2n^3)$ to one linear system of equations (4.38) of dimension n^3 and n linear systems of equations (4.39) of dimension n^2. For all these problems the method of separation of variables can be used.

After we find the solution of these problems we easily retrieve vectors $\mathbf{UI}^{(i,j,k)}$ by using the relations (4.36).

4.6. The Method of Separation of Variables

Let us consider the method of separation of variables for problems (4.38) and (4.39).

Problem (4.38) can be repressented in the form

$$C^{(3)}\mathbf{V} = \mathbf{f}, \qquad \mathbf{V},\mathbf{f} \in \mathrm{I\!R}^{(n^3)} \tag{4.40}$$

with the matrix

$$C^{(3)} = \frac{1}{2}C_0 \otimes I_0 \otimes I_0 + I_0 \otimes C_0 \otimes I_0 + I_0 \otimes I_0 \otimes C_0,$$

were I_0 is an $(n\times n)$-identity matrix and $(n\times n)$-matrix C_0 has the form

$$C_0 = \frac{1}{3}\begin{bmatrix} 3 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 3 \end{bmatrix}. \tag{4.41}$$

If we represent matrix C_0 in the form

$$C_0 = Q_0\Lambda_0 Q_0^T,$$

where Λ_0 is an $(n\times n)$-diagonal matrix and Q_0 is an $(n\times n)$-orthogonal matrix $(Q_0^{-1} = Q_0^T)$, then matrix $C^{(3)}$ can be rewritten in the form

$$C^{(3)} = Q^{(3)}\Lambda^{(3)}Q^{(3)},$$

where

$$Q^{(3)} = Q_0 \otimes Q_0 \otimes Q_0,$$

$$\Lambda^{(3)} = \frac{1}{2}\Lambda_0 \otimes I_0 \otimes I_0 + I_0 \otimes \Lambda_0 \otimes I_0 + I_0 \otimes I_0 \otimes \Lambda_0.$$

Remark. $Q^{(3)}$ is an $(n^3 \times n^3)$-orthogonal matrix and $\Lambda^{(3)}$ is an $(n^3 \times n^3)$-diagonal matrix.

Then we can use the following method of solving system (4.40):

$$\begin{aligned} &(1) \qquad \tilde{\mathbf{f}} = \left[Q^{(3)}\right]^T \mathbf{f}, \\ &(2) \quad \Lambda^{(3)}\mathbf{W} = \tilde{\mathbf{f}}, \\ &(3) \qquad \mathbf{V} = Q^{(3)}\mathbf{W}. \end{aligned} \tag{4.42}$$

Similarly, problem (4.39) can be rewritten in the form

$$C^{(2)}\mathbf{u} = \mathbf{b}, \qquad \mathbf{u}, \mathbf{b} \in \mathbb{R}^{(n^2)} \tag{4.43}$$

with the matrix

$$C^{(2)} = K_0 \otimes I_0 + I_0 \otimes K_0,$$

where $(n \times n)$-matrix K_0 has the form

$$K_0 = \frac{1}{3}\begin{bmatrix} 10 & -1 & & & \\ -1 & 9 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 9 & -1 \\ & & & -1 & 10 \end{bmatrix}. \tag{4.44}$$

Representing matrix K_0 in the form

$$K_0 = R_0 D_0 R_0^T,$$

where D_0 is an $(n \times n)$-diagonal matrix and R_0 is an $(n \times n)$-orthogonal matrix, we can rewrite matrix $C^{(2)}$ in the form

$$C^{(2)} = Q^{(2)}\Lambda^{(2)}Q^{(2)^T},$$

where $Q^{(2)} = R_0 \otimes R_0$ and $\Lambda^{(2)} = D_0 \otimes I_0 + I_0 \otimes D_0$. Then, for solving system (4.43) we will use the same method as (4.42):

$$\begin{aligned} &(1) \qquad \tilde{\mathbf{b}} = \left[Q^{(2)}\right]^T \mathbf{b}, \\ &(2) \quad \Lambda^{(2)}\mathbf{W} = \tilde{\mathbf{b}}, \\ &(3) \qquad \mathbf{u} = Q^{(2)}\mathbf{W}. \end{aligned} \tag{4.45}$$

4.7. Preconditioned Conjugate Gradient Method

We will solve system (2.9) by a preconditioned method of conjugate gradients in the following form:

$$\begin{gathered} \mathbf{u}_0 = 0, \qquad \mathbf{u}^{(k+1)} = \mathbf{u}^k - \frac{1}{P_k}\left[\tilde{B}^{-1}\boldsymbol{\xi}^k - d_{k-1}(\mathbf{u}^k - \mathbf{u}^{k-1})\right], \\ \boldsymbol{\xi}^k = A\mathbf{u}^k - \mathbf{f}, \qquad P_k = \frac{\|B^{-1}\boldsymbol{\xi}^k\|_A}{\|\boldsymbol{\xi}^k\|^2_{B^{-1}}} - d_{k-1}, \qquad d_k = P_k\frac{\|\boldsymbol{\xi}^{k+1}\|^2_{B^{-1}}}{\|\boldsymbol{\xi}^k\|^2_{B^{-1}}}, \\ k = 0, 1, \ldots, k_\varepsilon; \qquad k_{-1} = 0. \end{gathered} \tag{4.46}$$

It is well known that for a given accuracy ε ($\varepsilon \ll 1$) in the sense of inequality

$$\|\mathbf{u}^{k_\varepsilon+1} - \mathbf{u}^*\|_A \leq \varepsilon \|\mathbf{u}^\circ - \mathbf{u}^*\|_A, \tag{4.47}$$

where $\mathbf{u}^* = A^{-1}\mathbf{f}$ and $\mathbf{u}^\circ \in \mathbb{R}^N$ is any initial vector, the number of iterations K_ε can be chosen from the inequality

$$K_\varepsilon > \frac{\ell n\left(\frac{\varepsilon}{2}\right)}{\ell n\ q},$$

where $q = \frac{\sqrt{\nu}-1}{\sqrt{\nu}+1}$ (the value of ν is defined in (4.27)).

So, we have:

THEOREM 3. *The number of operations for solving system (2.9) by method (4.46) with matrix $\tilde{B}$ defined in (4.26) with accuracy ε in the sense of (4.47) is evaluated above by the expression $cN^{4/3}\ln\left(\frac{2}{\varepsilon}\right)$, where the constant $c > 0$ does not depend on N.*

Remark. If $\varepsilon = 10^{-6}$ then $K_\varepsilon \leq 60$ iterations.

5. Results of the Numerical Experiments

The method of preconditioning on the basis of multilevel substructuring as discussed above was tested on the model problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega = (0,1)^3 \subset \mathbb{R}^3, \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

with the nonconformal finite element method of approximation.

The domain was divided into n^3 cubes (n in each direction) and each cube was partitioned into 6 tetrahedra. The total dimension of the original algebraic system was $N = 12n^3 - 6n^2$.

The right hand side was generated randomly. The original algebraic problem has been solved by the conjugate gradient method in the form of (4.46) with the preconditioner in the form of (4.26) with accuracy $\varepsilon = 10^{-6}$. For comparison that problem has been solved by the same method without preconditioning. The condition number of matrix $B^{-1}A$ was calculated from the relation between conjugate gradients and Lanczos algorithm ([13]).

The method was implemented in FORTRAN-77 in DOUBLE PRECISION. All experiments were carried out on a Sun Workstation. The results are summarized in Table 1.

Table 1

n	N	CG Without Preconditioning			CG With Preconditioning		
		n_{iter}	cond	time (sec)	n_{iter}	cond	time (sec)
4	672	40	66	0.18	22	9.84	0.22
8	5760	73	265	2.18	24	10.7	1.27
16	47616	130	1062	49.2	24	11.94	15.7
32	387072	200<	—	1248	25	12.2	163
40	758400				25	12.26	376
50	1485000				25	12.33	771

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