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ON EFFICIENT TIME-STEPPING METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract—Efficient procedures for time-stepping Galerkin methods for approximating smooth solutions of quasilinear second-order hyperbolic equations are considered. The techniques presented can be used to analyze approximation procedures for related second-order-in-time quasilinear partial differential equations which have applications including initial-boundary value problems for vibrations (possibly) with inertia, dynamics of rotating fluids, and nonlinear viscoelasticity. The procedure involves the use of a preconditioned iterative method for approximately solving the different linear systems of equations arising at each time step in a discrete-time Galerkin method. Optimal order L^2 spatial errors and almost optimal order work estimates are obtained for the second-order hyperbolic case.

1. INTRODUCTION

We shall consider, as a model problem, efficient procedures for time-stepping Galerkin methods for approximating smooth solutions of quasilinear second-order hyperbolic equations. Equations of this type are generalized wave equations and are used as model equations for many different types of vibrational problems. We consider the problem of approximating the smooth solution $u = u(x, t)$ which satisfies

$$\begin{aligned} &\text{(a)}\ \ c(x)\frac{\partial^2 u}{\partial t^2} - \nabla \cdot [a(x, u)\nabla u] = f(x, t, u), \quad x \in \Omega, t \in J, \\ &\text{(b)}\ \ u(x, 0) = u_0(x), \quad x \in \Omega, \\ &\text{(c)}\ \ \frac{\partial u}{\partial t}(x, 0) = v_0(x), \quad x \in \Omega, \\ &\text{(d)}\ \ a(x, u)\frac{\partial u}{\partial \nu} = g(x, t), \quad x \in \partial\Omega, t \in J, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d \leq 3$, with boundary $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$, $J \equiv (0, T]$, and c, a, f, u_0, v_0, and g are prescribed. We shall first present a Crank-Nicolson–Galerkin approximation to (1.1) which produces a different linear system of equations to be solved at each time step. Procedures of this type have been analyzed in[1–4]. Our modification of the basic procedure will consist of using a preconditioned iterative procedure for only approximating the solution of these linear equations at each time step. The use of a preconditioning matrix eliminates the need to refactor a new matrix at each time step, while the iterative procedure is used to stabilize the resulting algorithm. Using this modification, we obtain the same order error estimates as for the base scheme with greatly reduced computational complexity. We obtain very nearly optimal possible work estimates for our procedure.

The techniques presented here can also be used to analyze approximation procedures for initial-boundary value problems for equations of the form

$$c(x)\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \left[\tilde{a}(x, \nabla u)\nabla u + b(x, \nabla u)\nabla \frac{\partial u}{\partial t}\right] = f(x, t, u, \nabla u), \quad x \in \Omega, \quad t \in J, \tag{1.2}$$

with appropriate initial and boundary conditions. Equations of this type have been used as models in nonlinear viscoelasticity and hydrodynamics. Existence, uniqueness, and stability of equations of this type have been studied by Dafermos, Greenberg, MacCamy, Mizel, Showalter and others[5–9]. The coefficient $\bar{a}$ can be allowed to degenerate to zero in (1.2).

We can also treat approximations of solutions of equations of the form

$$c(x)\frac{\partial^2 u}{\partial t^2} - \nabla\cdot\left[\bar{a}(x,\nabla u)\nabla u + \bar{b}(x,\nabla u)\nabla\frac{\partial u}{\partial t} + e(x,\nabla u)\nabla\frac{\partial^2 u}{\partial t^2}\right] = f(x,t,u,\nabla u),$$
$$x\in\Omega,\ t\in J, \tag{1.3}$$

with appropriate initial and boundary conditions. Equations of this type have been used as classical vibration models[10, §278] and in the dynamics of rotating fluids[11, 12]. The coefficients $\bar{a}$ or $\bar{b}$ can be allowed to degenerate to zero in (1.3).

Efficient time-stepping procedures of the type presented here have been used by the author and others, for pseudoparabolic equations in [13], for parabolic equations in [14, 15], and for systems of equations used to model miscible displacement in porous media in [16–18].

In Section 2 we introduce finite element spaces, present the hypotheses on (1.1) and its solution u, discuss an elliptic projection of u, and present various Crank–Nicolson–Galerkin methods for (1.1)–(1.3). In Section 3 we present our preconditioned modification of the base method and discuss the effect of the iterative stabilization on a single time step. We obtain global error estimates for both the base scheme and the iterative modification in Section 4. Section 5 contains a brief description of estimates of the computational complexity of the methods presented in the paper.

2. PRELIMINARIES AND DESCRIPTION OF GALERKIN METHODS

Let $(\varphi,\psi)=\int_\Omega \varphi\psi\,dx$, $\|\psi\|^2=(\psi,\psi)$, $\langle\varphi,\psi\rangle=\int_{\partial\Omega}\varphi\psi\,ds$, and $|\varphi|^2=\langle\varphi,\varphi\rangle$. Let $W_s^k(\Omega)$ be the Sobolev space on Ω with norm

$$\|\psi\|_{W_s^k}=\left(\sum_{|\alpha|\le k}\left\|\frac{\partial^\alpha\psi}{\partial x^\alpha}\right\|_{L^s(\Omega)}\right)^{1/s},$$

with the usual modification for $s=\infty$. When $s=2$, let $\|\psi\|_{W_2^k}=\|\psi\|_{H^k}=\|\psi\|_k$. If $\nabla F=(F_1,F_2)$, write $\|\nabla F\|_{W_s^k}$ in place of $(\|F_1\|^s_{W_s^k}+\|F_2\|^s_{W_s^k})^{1/s}$. Let $H^s(\partial\Omega)$ denote the corresponding Sobolev space on $\partial\Omega$ with norm $\|\psi\|_{H^s(\partial\Omega)}\equiv|\psi|_s$ (with $|\psi|\equiv|\psi|_0$).

Let $\{\mathcal{M}_h\}$ be a family of finite-dimensional subspaces of $H^1(\Omega)$ with the following property:

For $p=2$ or $p=\infty$, there exist an integer $r\ge 2$ and a constant K_0 such that, for $1\le q\le r$ and $\psi\in W_p^q(\Omega)$,

$$\inf_{\chi\in\mathcal{M}_h}\{\|\psi-\chi\|_{W_p^0}+h\|\psi-\chi\|_{W_p^1}\}\le K_0\|\psi\|_{W_p^q}h^q. \tag{2.1}$$

We also assume that the family $\{\mathcal{M}_h\}$ satisfies the following so-called "inverse hypotheses": if $\psi\in\mathcal{M}_h$,

$$\text{(a)}\ \|\psi\|_{L^\infty(\Omega)}\le K_0h^{-(d/2)}\|\psi\|,$$
$$\text{(b)}\ \|\psi\|_1\le K_0h^{-1}\|\psi\|. \tag{2.2}$$

Restrict Ω as follows (with (S) denoting the collection of restrictions):

(S)
(1) The Neumann problem for $-\Delta+I$ on Ω is H^2-regular.
(2) $\partial\Omega$ is Lipschitz.

Assume the following regularity for c, a, $\bar{a}$, b, e, f and u:

(Q) 1. There exist uniform constants such that

$$\text{(a)}\ 0<a_*\le a(x,u)\le a^*\le K_1,$$
$$\text{(b)}\ 0\le\bar{a}_*\le\bar{a}(x,\nabla u)\le\bar{a}^*\le K_1,$$

$$
\begin{aligned}
&\text{(c)}\ 0 < c_* \le c(x) \le c^* \le K_1,\\
&\text{(d)}\ 0 < b_* \le b(x, \nabla u) \le K_1,\\
&\text{(e)}\ 0 \le \tilde{b}_* \le \tilde{b}(x, \nabla u) \le K_1,\\
&\text{(f)}\ 0 < e_* \le e(x, \nabla u) \le K_1,\\
&\text{(g)}\ |f(x, t, u)| \le K_1.
\end{aligned}
\tag{2.3}
$$

2. The functions $a = a(x, u)$, $\tilde{a} = \tilde{a}(x, q_1, q_2)$, $b = b(x, q_1, q_2)$, $\tilde{b} = \tilde{b}(x, q_1, q_2)$, $e = e(x, q_1, q_2)$, and $f = f(x, t, u)$ are continuously differentiable with respect to u (respectively ∇u) and have a uniform bound, K_1, satisfying (for $i = 1, 2$)

$$
\begin{aligned}
&|a|, |\tilde{a}|, |b|, |\tilde{b}|, |c|, |e|, |f|, \left|\frac{\partial a}{\partial u}\right|, \left|\frac{\partial \tilde{a}}{\partial q_i}\right|, \left|\frac{\partial b}{\partial q_i}\right|,\\
&\left|\frac{\partial e}{\partial q_i}\right|, \left|\frac{\partial f}{\partial u}\right|, \left|\frac{\partial^2 a}{\partial u^2}\right|, \left|\frac{\partial^2 \tilde{a}}{\partial q_i^2}\right| \le K_1.
\end{aligned}
\tag{2.4}
$$

Define

$$
\|\psi\|_{W_p{}^q((a,b);X)} \equiv \| \ \|\psi(\cdot, t)\|_X \|_{W_p{}^q(a,b)}, \quad 1 \le p, q \le \infty. \tag{2.5}
$$

Let u, the solution of (1.1) satisfy the following regularity assumptions:
R:

$$
\begin{aligned}
&\|u\|_{L^\infty(J;H^r)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(J;H^r)} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(J;H^r)} \le K_2,\\
&\left\|u\right\|_{L^\infty(J;H^3)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J;H^2)} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^\infty(J;H^2)} \le K_2,\\
&\left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^2(J;L^2)} + \left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^1(J;H^1)} + \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^2(J;L^2)} \le K_2.
\end{aligned}
\tag{2.6a–c}
$$

Similar regularity assumptions must be satisfied by the solutions of (1.2) and (1.3) but we shall not make these explicit here.

As in [19], we shall introduce an auxiliary elliptic problem to aid in our analysis. Define W in $\mathcal{M}_h$ to be the unique function which, for $t \in [0, T]$, satisfies

$$
(a(u)\nabla W, \nabla\chi) + (W, \chi) = (a(u)\nabla u, \nabla\chi) + (u, \chi), \quad \chi \in \mathcal{M}_h. \tag{2.7}
$$

Then as in [1, 19, 20] we obtain the following lemma.

LEMMA 2.1

There exists a constant $K_3 = K_3(\Omega, a_*, K_0, K_1, K_2)$ such that for $2 \le q \le r$, $\eta = u - W$, and $s = 0$ or 1,

$$
\begin{aligned}
&\text{(a)}\ \|\eta\|_{L^\infty(J;H^s)} \le K_3 h^{q-s} \|u\|_{L^\infty(J;H^q)},\\
&\text{(b)}\ \left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(J;H^s)} + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^2(J;H^s)} \le K_3 h^{q-s}\left\{\|u\|_{L^2(J;H^q)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(J;H^q)} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(J;H^q)}\right\}.
\end{aligned}
\tag{2.8}
$$

We also make the assumption on $\{\mathcal{M}_h\}$ and u that there exists a constant K_4 such that

$$
\|W\|_{L^\infty(J;L^\infty)} + \|\nabla W\|_{L^\infty(J;L^\infty)} + \left\|\frac{\partial W}{\partial t}\right\|_{L^\infty(J;L^\infty)} + \left\|\nabla \frac{\partial W}{\partial t}\right\|_{L^2(J;L^\infty)} \le K_4. \tag{2.9}
$$

Sufficient conditions for (2.9) to hold can be found in [14, 19]. Also as in [1, 14, 20] we can obtain the following lemma.

LEMMA 2.2
There exists a constant $K_5 = K_5(\Omega, a_*, K_0, K_1, K_2)$ such that

$$\left\|\frac{\partial^2 W}{\partial t^2}\right\|_{L^\infty(J:H^1)} + \left\|\frac{\partial^3 W}{\partial t^3}\right\|_{L^\infty(J:H^1)} \leq K_5. \tag{2.10}$$

We shall consider discrete-time Galerkin approximations. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbf{Z}$, and $t^\sigma = \sigma\Delta t$, $\sigma \in \mathbf{R}$. Also let $\psi^n \equiv \psi^n(x) \equiv \psi(x, t^n)$, and

$$\begin{aligned} &\text{(a)}\ d_t\psi^n = \frac{\psi^{n+1} - \psi^n}{\Delta t}, \\ &\text{(b)}\ d_t^2\psi^n = \frac{\psi^{n+1} - 2\psi^n + \psi^{n-1}}{(\Delta t)^2}. \end{aligned} \tag{2.11}$$

We shall consider Crank–Nicolson–Galerkin methods for our base time-stepping procedure for each of the equations. Let U: $\{t_0, \ldots, t_N\} \mapsto \mathcal{M}_h$ be an approximation to the solution of (1.1). Assuming that U^k are known for $k \leq n$, determine U^{n+1} by

$$(cd_t^2 U^n, \chi) + \left(a(U^n)\nabla\left(\frac{U^{n+1} + U^{n-1}}{2}\right), \nabla\chi\right) = (f(t^n, U^n), \chi) + \langle g(t^n), \chi\rangle, \quad \chi \in \mathcal{M}_h. \tag{2.12}$$

Similarly, we define our approximation to the solution of (1.2) by

$$\begin{aligned} (cd_t^2 U^n, \chi) + \left(\bar{a}(\nabla U^n)\nabla\left(\frac{U^{n+1} + U^{n-1}}{2}\right), \nabla\chi\right) + \left(b(\nabla U^n)\nabla\frac{U^{n+1} - U^{n-1}}{2\Delta t}, \nabla\chi\right) \\ = (f(t^n, U^n, \nabla U^n), \chi) + \langle g(t^n), \chi\rangle, \quad \chi \in \mathcal{M}_h, \end{aligned} \tag{2.13}$$

and our approximation to the solution of (1.3) by

$$\begin{aligned} (cd_t^2 U^n, \chi) + \left(\bar{a}(\nabla U^n)\nabla\left(\frac{U^{n+1} + U^{n-1}}{2}\right), \nabla\chi\right) + \left(\bar{b}(\nabla U^n)\nabla\frac{U^{n+1} - U^{n-1}}{2\Delta t}, \nabla\chi\right) \\ + (e(\nabla U^n)\nabla d_t^2 U^n, \nabla\chi) = (f(t^n, U^n, \nabla U^n), \chi) + \langle g(t^n), \chi\rangle, \quad \chi \in \mathcal{M}_h. \end{aligned} \tag{2.14}$$

3. ITERATIVE PROCEDURES

In this section, we shall present the linear equations arising from (2.12)–(2.14). We note that in each case, the coefficient matrices change with each time step. In order to avoid factorization of different matrices at each time step for the solution of the linear equations, we shall discuss an iterative method for approximating their solution. The analysis presented here will extend the analysis of [13, 14] to the eqns (1.1)–(1.3).

Let $\{\mu_i\}_{i=1}^M$ be a basis for $\mathcal{M}_h$. Let U^m from (2.12) be written as

$$U^m = \sum_{i=1}^M \xi_i^m \mu_i. \tag{3.1}$$

We then see that using (3.1), (2.12) can be written as

$$\left[C + \frac{(\Delta t)^2}{2} A^n(\xi)\right](\xi^{n+1} - \xi^n) = C(\xi^n - \xi^{n-1}) - \frac{(\Delta t)^2}{2} A^n(\xi)(\xi^n + \xi^{n-1}) + (\Delta t)^2 F_1{}^n(\xi) \equiv R_1(\xi) \tag{3.2}$$

where

$$\text{(a) } C = ((c\mu_j, \mu_i)),$$
$$\text{(b) } A^n(\xi) = \left(\left(a\left(\sum_{l=1}^{M} \xi_l^n \mu_l\right)\nabla\mu_j, \nabla\mu_i\right)\right), \text{ and} \tag{3.3}$$
$$\text{(c) } F_1^n(\xi) = \left(\left(f\left(t^n, \sum_{l=1}^{M} \xi_l^n \mu_l\right), \mu_i\right) + \langle g(t^n), \mu_i\rangle\right).$$

Similarly, (2.13) can be written as

$$\left[C + \frac{(\Delta t)^2}{2} A^n(\xi) + \frac{\Delta t}{2} B^n(\xi)\right](\xi^{n+1} - \xi^n) = \left[C - \frac{\Delta t}{2} B^n(\xi)\right](\xi^n - \xi^{n-1}) - \frac{(\Delta t)^2}{2} A^n(\xi)(\xi^n + \xi^{n-1}) + (\Delta t)^2 F_2^n(\xi) \tag{3.4}$$

and (2.14) can be written as

$$\left[C + E^n(\xi) + \frac{(\Delta t)^2}{2} A^n(\xi) + \frac{\Delta t}{2} B^n(\xi)\right](\xi^{n+1} - \xi^n) = \left[C + E^n(\xi) - \frac{\Delta t}{2} B^n(\xi)\right](\xi^n - \xi^{n-1}) - \frac{(\Delta t)^2}{2} A^n(\xi)(\xi^n + \xi^{n-1}) + (\Delta t)^2 F_2^n(\xi) \tag{3.5}$$

where B^n and E^n are defined as in (3.3.b) with the coefficient a replaced by b and e respectively and F_2^n is defined in an analogous manner to F_1^n.

Note that since the matrices A^n, B^n and E^n change with time, straightforward solution of (3.2), (3.4) or (3.5) would involve the factorization of new matrices at each time step. Instead of solving (3.2) exactly, we shall approximate the solution by using an iterative procedure which has been preconditioned by

$$L^0 = C + \frac{(\Delta t)^2}{2} A^0(\xi). \tag{3.6}$$

Similarly, for (3.4) and (3.5) we shall precondition with

$$\tilde{L}^0 = C + \frac{(\Delta t)^2}{2} A^0(\xi) + \frac{\Delta t}{2} B^n(\xi)$$

and

$$\hat{L}^0 = C + E^0(\xi) + \frac{(\Delta t)^2}{2} A^0(\xi) + \frac{\Delta t}{2} B^0(\xi),$$

respectively. The preconditioning process eliminates the need for factoring new matrices at each time step, while the iterative procedure stabilizes the resulting problem. The stabilization process requires iteration only until a predetermined norm reduction is achieved.

Let the approximation of U^n from (2.12) produced by only approximately solving (3.2) using the preconditioner (3.6) be denoted by

$$V^m = \sum_{i=1}^{M} \gamma_i^m \mu_i. \tag{3.7}$$

A starting procedure for determining V^0 and V^1 will be discussed later. Assuming that these quantities are known, we shall determine γ^{n+1}, $n \geq 1$, using a preconditioned iterative method to approximate ξ^{n+1} from (3.2). As an initial guess for $\xi^{n+1} - \xi^n$ for $n \geq 2$, we shall use quadratic extrapolation. Specifically, we shall use

$$X_0 = 2\gamma^n - 3\gamma^{n-1} + \gamma^{n-2} \tag{3.8}$$

as the initialization for the iterative procedure for $\gamma^{n+1} - \gamma^n$. Since we use γ^n, γ^{n-1} and γ^{n-2} in the coefficient matrices to determine γ^{n+1}, the errors in the approximate solution will accumulate.

In order to estimate the cumulative error, we first consider the single step error. Define $\bar{\gamma}^{n+1}$ to satisfy

$$L^n(\gamma)(\bar{\gamma}^{n+1} - \gamma^n) = \left[C + \frac{(\Delta t)^2}{2} A^n(\gamma)\right](\bar{\gamma}^{n+1} - \gamma^n) = R_1(\gamma), \quad n \geq 1, \tag{3.9}$$

from (3.2). For all of the analysis to follow, we can use any preconditioned iterative method which yields norm reductions of the form

$$\|L^n(\gamma)^{1/2}(\bar{\gamma}^{n+1} - \gamma^{n+1})\|_e \leq \rho_1 \|L^n(\gamma)^{1/2}(\bar{\gamma}^{n+1} - 3\gamma^n + 3\gamma^{n-1} - \gamma^{n-2})\|_e, \tag{3.10}$$

where $0 < \rho_1 < 1$ and the subscript indicates the Euclidean norm of the vector. A particularly efficient iterative procedure for obtaining (3.10) is the preconditioned conjugate gradient method presented in [13, 14, 24, 25, 26].

Let

$$\begin{aligned} &\text{(a)}\ \|\varphi\|_c^2 \equiv (c\varphi, \varphi) \\ &\text{(b)}\ \|\varphi\|_{a^n}^2 \equiv \left(\frac{1}{2} a(V^n)\nabla\varphi, \nabla\varphi\right) \\ &\text{(c)}\ |||\varphi|||_n = \|\varphi\|_c + \Delta t\|\varphi\|_{a^n}, \end{aligned} \tag{3.11}$$

be special norms and semi-norms. Note that $\|\cdot\|_{a^n}$ are uniformly equivalent to $\|\nabla \cdot\|$. Then letting

$$\bar{V}^m = \sum_{i=1}^{M} \bar{\gamma}_i \mu_i, \tag{3.12}$$

we see that $\bar{V}^{n+1}$ satisfies

$$\begin{aligned} &\left(c\frac{\bar{V}^{n+1} - 2V^n + V^{n-1}}{(\Delta t)^2}, \chi\right) + \left(a(V^n)\nabla\left(\frac{\bar{V}^{n+1} + V^{n-1}}{2}\right), \nabla\chi\right) \\ &\qquad = (f(t^n, U^n), \chi) + \langle g(t^n), \chi\rangle, \quad \chi \in \mathcal{M}_h. \end{aligned} \tag{3.13}$$

We also see that, using (3.11), (3.10) can be written as

$$|||\bar{V}^{n+1} - V^{n+1}|||_n \leq \rho_1' |||\delta^3 V^n|||_n, \quad n \geq 2, \tag{3.14}$$

where

$$\begin{aligned} &\text{(a)}\ \rho_1' = \frac{\rho_1}{1-\rho_1}, \\ &\text{(b)}\ \delta\varphi^n \equiv \varphi^{n+1} - \varphi^n, \\ &\text{(c)}\ \delta^2\varphi^n \equiv \varphi^{n+1} - 2\varphi^n + \varphi^{n-1}, \\ &\text{(d)}\ \delta^3\varphi^n \equiv \varphi^{n+1} - 3\varphi^n + 3\varphi^{n-1} - \varphi^{n-2}. \end{aligned} \tag{3.15}$$

We next discuss a starting procedure for obtaining V^0, V^1 and V^2. We shall follow the ideas of [3] in determining V^0 and V^1. Let $V^0 = W(0)$; i.e. project u_0 into $\mathcal{M}_h$. This will require the factorization of one additional matrix to solve the elliptic problem (2.7). Then approximate $u(x, \Delta t)$ by

$$u^* = u(x,0) + \Delta t \frac{\partial u}{\partial t}(x,0) + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}(x,0).$$

Project u^* into $\mathcal{M}_h$. The derivative $(\partial^2 u/\partial t^2)$ is evaluated using the differential equations. The solution of the second elliptic problem can be approximated using the factored matrix used to determine V^0. We can thus obtain the estimate

$$\|\nabla(V-W)^0\| + \|\nabla(V-W)^1\| + \|d_t(V-W)^0\| \le C\{h^r + (\Delta t)^2\}. \tag{3.16}$$

Once we have V^0 and V^1 satisfying (3.16), V^2 can be determined using the same preconditioned iterative procedure as above by initializing the iterative procedure by $X_0 = \gamma^1 - \gamma^0$. For details of a starting procedure using the iterative procedure, see [14].

4. *A PRIORI* ERROR ESTIMATES

In this section we develop *a priori* bounds for the errors $U^n - u^n$ and $V^n - u^n$ for the procedures defined in (2.12) and (3.13) respectively. Similar results yielding optimal order H^1-estimates can be obtained using similar techniques for the procedures defined in (2.13) and (2.14) and their iterative counterparts. Theorem 4.1 yields optimal order L^2-estimates for the procedure satisfying (2.12) and (3.16) under restrictions given in (4.18). Under the slightly stronger mesh-ratio restriction

$$\Delta t \le C^* h, \tag{4.1}$$

we obtain optimal order L^2-estimates for the iterative procedure satisfying (3.13) and (3.16) in Theorem 4.2.

THEOREM 4.1

Let S, Q, R, and the restrictions on $\{\mathcal{M}_h\}$ of Section 2 hold. Let U^n satisfy (2.12) and (3.16). Then there exist constants τ, h_0, and $K_6 = K_6(K_i; i = 0, \dots, 5)$ such that if $r > (d/2)$, $\Delta t \le \tau$, $h \le h_0$, and $\Delta t < h^{d/4}$,

$$\sup_{t^n}\{\|u - U\| + h\|u - U\|_1\} \le K_6\{h^r + (\Delta t)^2\}. \tag{4.2}$$

Proof

Let $\eta^n = u^n - W^n$ and $\zeta^n = U^n - W^n$. From (1.1), (2.7) and (2.12), we see that

$$(cd_t^2\zeta^n, \chi) + \left(a(U^n)\nabla\frac{\zeta^{n+1}+\zeta^{n-1}}{2}, \nabla\chi\right) = \left(c\left[\frac{\partial^2 u}{\partial t^2} - d_t^2 W^n\right], \chi\right) + (\eta^n, \chi)$$
$$+ \left(a(u^n)\nabla W^n - a(U^n)\nabla\frac{W^{n+1}+W^{n-1}}{2}, \nabla\chi\right) + (f(t^n, U^n) - f(t^n, u^n), \chi), \quad \chi \in \mathcal{M}_h. \tag{4.3}$$

We shall let $\chi = \zeta^{n+1} - \zeta^{n-1} = \Delta t(d_t\zeta^n + d_t\zeta^{n-1})$ in (4.3). Using this test function and (3.11), the left hand side of (4.3) becomes

$$\left(c\frac{d_t\zeta^n - d_t\zeta^{n-1}}{\Delta t}, \Delta t(d_t\zeta^n + d_t\zeta^{n-1})\right) + \left(a(U^n)\nabla\frac{\zeta^{n+1}+\zeta^{n-1}}{2}, \nabla(\zeta^{n+1} - \zeta^{n-1})\right)$$
$$= \|d_t\zeta^n\|_c^2 - \|d_t\zeta^{n-1}\|_c^2 + \frac{1}{2}\{\|\zeta^{n+1}\|_{a^n}^2 - \|\zeta^{n-1}\|_{a^n}^2\}. \tag{4.4}$$

In order to obtain telescoping sums when (4.4) is summed on n, we must shift the indices in two of the terms above. Note that

$$\|\zeta^{n-1}\|^2_{a^n} = \|\zeta^{n-1}\|^2_{a^{n-2}} + ([a(U^n) - a(U^{n-2})]\nabla\zeta^{n-1}, \nabla\zeta^{n-1})$$
$$= \|\zeta^{n-1}\|^2_{a^{n-2}} + \left(\frac{\partial a}{\partial u}[\delta\zeta^{n-1} + \delta\zeta^{n-2} + \delta W^{n-1} + \delta W^{n-2}]\nabla\zeta^{n-1}, \nabla\zeta^{n-1}\right)$$
$$\leq \|\zeta^{n-1}\|^2_{a^{n-2}} + C[\|\delta\zeta^{n-1}\|_{L^\infty} + \|\delta\zeta^{n-2}\|_{L^\infty} + \Delta t]\|\zeta^{n-1}\|^2_{a^{n-2}}. \tag{4.5}$$

We shall henceforth use C as a generic constant in our analysis. For the first term on the right of (4.3), we obtain

$$\left|\sum_{n=1}^{l-1}\left(c\left[\frac{\partial^2 u}{\partial t^2} - d_t^2 W^n\right], \Delta t(d_t\zeta^n + d_t\zeta^{n-1})\right)\right|$$
$$\leq C\sum_{n=0}^{l-1}\{\|d_t^2\eta^n\|^2 + \|d_t\zeta^n\|_c^2\}\Delta t + C(\Delta t)^4. \tag{4.6}$$

We next bound the second and fourth terms on the right of (4.3) as follows

$$\left|\sum_{n=1}^{l-1}(\eta^n + f(t^n, U^n) - f(t^n, u^n), \Delta t(d_t\zeta^n + d_t\zeta^{n-1}))\right|$$
$$\leq C\sum_{n=0}^{l-1}\{\|\eta^n\|^2 + \|\zeta^n\|_c^2 + \|d_t\zeta^n\|_c^2\}\Delta t. \tag{4.7}$$

We split the third term on the right side of (4.3) as follows

$$\left|\sum_{n=1}^{l-1}\left(a(u^n)\nabla\left(W^n - \frac{W^{n+1} + W^{n-1}}{2}\right) + [a(u^n) - a(U^n)]\nabla\frac{W^{n+1} + W^{n-1}}{2}, \nabla\chi\right)\right|$$
$$= \left|\sum_{n=1}^{l-1}(T_1 + T_2, \nabla\chi)\right|. \tag{4.8}$$

In order to treat the terms in (4.8), we shall sum by parts in time,

$$\left|\sum_{n=1}^{l-1}(T_1, \nabla[(\zeta^{n+1} + \zeta^n) - (\zeta^n + \zeta^{n-1})])\right|$$
$$\leq \left|\sum_{n=2}^{l-1}\left([a(u^n) - a(u^{n-1})]\nabla\left(W^n - \frac{W^{n+1} + W^{n-1}}{2}\right), \nabla(\zeta^n + \zeta^{n-1})\right)\right|$$
$$+ \left|\sum_{n=2}^{l-1}\left(a(u^{n-1})\nabla\left[\left\{W^n - \frac{W^{n+1} + W^{n-1}}{2}\right\} - \left\{W^{n-1} - \frac{W^n + W^{n-2}}{2}\right\}\right], \nabla(\zeta^n + \zeta^{n-1})\right)\right| \tag{4.9}$$
$$+ \left|(a(u^{l-1})\nabla\left(\frac{1}{2}\delta^2 W^{l-1}\right), \nabla(\zeta^l + \zeta^{l-1}))\right| + \left|(a(u^1)\nabla\left(\frac{1}{2}\delta^2 W^1\right), \nabla(\zeta^1 + \zeta^0))\right|$$
$$\leq C\left\{\sum_{n=1}^{l-1}\|\zeta^n\|^2_{a^{n-1}}\Delta t + (\Delta t)^4\right\}$$
$$+ \frac{1}{8}\{\|\zeta^l\|^2_{a^{l-1}} + \|\zeta^{l-1}\|^2_{a^{l-2}} + \|\zeta^1\|^2_{a^0} + \|\zeta^0\|^2_{a^0}\}.$$

Similarly, we see that

$$\left|\sum_{n=1}^{l-1}(T_2, \nabla[(\zeta^{n+1} + \zeta^n) - (\zeta^n + \zeta^{n-1})])\right|$$
$$\leq \left|\sum_{n=2}^{l-1}\left([a(u^n) - a(U^n)]\nabla\left\{\frac{W^{n+1} + W^{n-1}}{2} - \left(\frac{W^n + W^{n-2}}{2}\right)\right\}, \nabla(\zeta^n + \zeta^{n-1})\right)\right|$$

$$
\begin{aligned}
&+\left|\sum_{n=2}^{l-1}\left([a(u^n)-a(U^n)-\{a(u^{n-1})-a(U^{n-1})\}]\nabla\frac{W^{n+1}+W^{n-1}}{2},\nabla(\zeta^n+\zeta^{n-1})\right)\right| \\
&\quad+\left|\left([a(u^{l-1})-a(U^{l-1})]\nabla\frac{W^l+W^{l-2}}{2},\nabla(\zeta^l+\zeta^{l-1})\right)\right| \\
&\quad+\left|\left([a(u^1)-a(U^1)]\nabla\frac{W^2+W^0}{2},\nabla(\zeta^1+\zeta^0)\right)\right| \\
&\leq C\left(\sum_{n=1}^{l-1}\left[\|\zeta^n\|^2_{a^{n-1}}+\|\eta^n\|^2+\|d_t\eta^n\|^2+\|d_t\zeta^n\|^2_c\right]\Delta t+(\Delta t)^4\right\}.
\end{aligned}
\tag{4.10}
$$

Combining (4.3)–(4.10), we see that after summing (4.3) on n for $n = 1$ to $n = l - 1$, we use Lemma 2.1 to obtain

$$
\begin{aligned}
&\|d_t\zeta^{l-1}\|^2_c+\frac{1}{4}\{\|\zeta^l\|^2_{a^{l-1}}+\|\zeta^{l-1}\|^2_{a^{l-2}}\} \\
&\leq C\sum_{n=0}^{l-1}\|\delta\zeta^{n-1}\|_{L^\infty}\{\|d_t\zeta^{n-1}\|^2_c+\|\zeta^n\|^2_{a^{n-1}}+\|\zeta^{n-1}\|^2_{a^{n-2}}\} \\
&\quad+C\sum_{n=0}^{l-1}\{\|d_t\zeta^n\|^2_c+\|\zeta^n\|^2_{a^{n-1}}+\|\zeta^n\|^2_c\}\,\Delta t \\
&+C_1\{\|\zeta^l\|^2_c+\|\zeta^{l-1}\|^2_c\}+C\{\|\zeta^0\|^2_{a^0}+\|\zeta^1\|^2_{a^0}+\|d_t\zeta^0\|^2_c+h^{2r}+(\Delta t)^4\}.
\end{aligned}
\tag{4.11}
$$

In order to bound the terms multiplied by C_1 on the right side of (4.11) and to introduce an L^2 term on the left hand side of (4.11), we note that

$$
\|\zeta^{n+1}\|^2_c-\|\zeta^n\|^2_c=2\Delta t(cd_t\zeta^n,\zeta^n)+(\Delta t)^2\|d_t\zeta^n\|^2_c\leq 2\Delta t\|d_t\zeta^n\|^2_c+\Delta t\|\zeta^n\|^2_c. \tag{4.12}
$$

Sum this inequality from $n = 1$ to the upper limits $l - 1$ and $l - 2$; then multiply the resulting inequalities by $C_1 + (1/4)$, add them to (4.11) and use (3.16) to obtain

$$
\begin{aligned}
&\|d_t\zeta^{l-1}\|^2_c+\frac{1}{4}\{\|\zeta^l\|^2_{a^{l-1}}+\|\zeta^{l-1}\|^2_{a^{l-2}}+\|\zeta^l\|^2_c\}\leq C\{h^{2r}+(\Delta t)^4\} \\
&\quad+\sum_{n=1}^{l-1}\|\delta\zeta^{n-1}\|_{L^\infty}[\|d_t\zeta^{n-1}\|^2_c+\|\zeta^n\|^2_{a^{n-1}}+\|\zeta^{n-1}\|^2_{a^{n-2}}] \\
&\quad+\sum_{n=0}^{l-1}\Delta t[\|d_t\zeta^n\|^2_c+\|\zeta^n\|^2_{a^{n-1}}+\|\zeta^n\|^2_c]\}.
\end{aligned}
\tag{4.13}
$$

In order to apply the discrete Gronwall lemma to (4.13), we wish to show that there exists a constant $C_0 > 0$ such that

$$
\sum_{n=0}^{l-2}\|\delta\zeta^n\|_{L^\infty}\leq C_0. \tag{4.14}
$$

The given starting procedure yields

$$
\|\delta\zeta^0\|_{L^\infty}\leq C_2. \tag{4.15}
$$

We shall use an induction argument as in [13, 14, 21] to yield (4.14) with the summation starting at $n = 1$. For $l = 2$, the inequality (4.13) and the estimate (3.16) imply that

$$
\|d_t\zeta^1\|^2_c\leq C\{\|d_t\zeta^0\|^2_c+\|\zeta^1\|^2_{a^0}+\|\zeta^0\|^2_{a^0}+h^{2r}+(\Delta t)^4\}\leq C\{h^{2r}+(\Delta t)^4\}. \tag{4.16}
$$

Then we have by (2.2.a), (4.15) and (4.16),

$$\|\delta\zeta^1\|_{L^\infty}+\|\delta\zeta^0\|_{L^\infty}\le C\Delta t\|d_t\zeta^1\|_c h^{-(d/2)}+C_2\le C\Delta t h^{-(d/2)}\{h^r+(\Delta t)^2\}+C_2. \tag{4.17}$$

Then if

$$r>\frac{d}{2}, \tag{4.18a}$$

and

$$\Delta t<h^{d/4}, \tag{4.18b}$$

we see that for Δt and h sufficiently small, (4.14) is satisfied with $l=3$. Assume the following induction hypothesis:

$$\sum_{n=0}^{k}\|\delta\zeta^n\|_{L^\infty}\le C_0 \quad \text{for} \quad 1\le k\le l-2. \tag{4.19}$$

We can now apply the discrete Gronwall lemma to (4.13) and obtain for $1\le l\le N$,

$$\|\zeta^l\|^2+\|\nabla\zeta^l\|^2+\|d_t\zeta^{l-1}\|^2\le \bar{C}\{h^{2r}+(\Delta t)^4\}. \tag{4.20}$$

Note that from (2.2.a) and (4.20),

$$\sum_{n=0}^{l-1}\|\delta\zeta^n\|_{L^\infty}\le\sum_{n=0}^{l-1}\Delta t\|d_t\zeta^n\|K_0h^{-(d/2)}\le\sum_{n=0}^{l-1}\Delta tK_0h^{-(d/2)}\bar{C}\{h^{2r}+(\Delta t)^4\}$$
$$\le TK_0\bar{C}h^{-(d/2)}\{h^{2r}+(\Delta t)^4\}. \tag{4.21}$$

Then if (4.18) is satisfied and Δt and h are sufficiently small, our induction argument is completed. Then since (4.21) holds for $1\le l\le N$, using (4.21), (2.8) and the triangle inequality, we obtain the desired result (4.2).

We shall next obtain the same order asymptotic error estimates as derived in Theorem 4.1 for the approximation V defined in Section 3. We shall see in Section 5 that the work estimates for the approximation V are far superior to those for the approximation U analyzed above.

THEOREM 4.2

Let S, Q, R, and the restrictions on $\{\mathcal{M}_h\}$ of Section 2 hold. Let V^n satisfy (3.13), (3.16) and (3.14) where

$$\rho_1'\le\left\{28\left[1+\frac{a^*K_0C^*}{2c_*}\right]^2\right\}^{-1}\Delta t.$$

Then there exist constants τ, h_0 and $K_7=K_7(C^*, K_i, i=0,\dots,5)$ such that if $r>(d/2)$, $\Delta t\le\tau$, $h\le h_0$, and $\Delta t\le\min\{h^{d/4}, C^*h\}$,

$$\sup_{t^n}\{\|u-V\|+h\|u-V\|_1\}\le K_7\{h^r+(\Delta t)^2\}. \tag{4.22}$$

Proof

Let $Z^n=V^n-W^n$ and η^n be as above. From (1.1), (2.7) and (3.13), we see that

$$(cd_t^2Z^n,\chi)+\left(a(V^n)\nabla\frac{Z^{n+1}+Z^{n-1}}{2},\nabla\chi\right)=\left(c\left[\frac{\partial^2u}{\partial t^2}-d_t^2W^n\right],\chi\right)+(\eta^n,\chi)$$
$$+\left(a(u^n)\nabla W^n-a(V^n)\nabla\frac{W^{n+1}+W^{n-1}}{2},\nabla\chi\right)+(f(t^n,V^n)-f(t^n,u^n),\chi)$$
$$+\left(c\frac{Z^{n+1}-\bar{Z}^{n+1}}{(\Delta t)^2},\chi\right)+(a(V^n)\frac{1}{2}\nabla(Z^{n+1}-\bar{Z}^{n+1}),\nabla\chi),\quad \chi\in\mathcal{M}_h. \tag{4.23}$$

We note that except for the last two terms on the right of (4.23), equation (4.23) corresponds exactly with (4.3). We must thus only show how the last two terms on the right of (4.23) are bounded. From (4.1), (3.11) and (3.14) we see that

$$
\begin{aligned}
|T_L| \equiv \Big| &\Big(c\,\frac{Z^{n+1}-\tilde{Z}^{n+1}}{(\Delta t)^2},\ \Delta t(d_t\zeta^n - d_t\zeta^{n-1})\Big) \\
&+ \Big(\frac{a(V^n)}{2}\nabla(Z^{n+1}-\tilde{Z}^{n+1}),\ \nabla(d_t\zeta^n - d_t\zeta^{n-1})\Big)\Delta t\Big| \\
\le \frac{1}{\Delta t}\,|||Z^{n+1}-\tilde{Z}^{n+1}|||_n &\{\|d_t\zeta^n\|_c + \|d_t\zeta^{n-1}\|_c + \Delta t[\|d_t\zeta^n\|_{a^n} + \|d_t\zeta^{n-1}\|_{a^n}]\} \\
&\le \frac{\rho_1'}{\Delta t}\Big[1+\frac{a^*K_0C^*}{2c_*}\Big][\|d_t\zeta^n\|_c + \|d_t\zeta^{n-1}\|_c]|||\delta^3V^n|||_n \\
&\le \rho_1'\Big[1+\frac{a^*K_0C^*}{2c_*}\Big]^2[\|d_t\zeta^n\|_c + \|d_t\zeta^{n-1}\|_c] \\
&+[\|d_t\zeta^n\|_c + \|d_t\zeta^{n-1}\|_c + \|d_t\zeta^{n-2}\|_c + C(\Delta t)^2].
\end{aligned}
\tag{4.24}
$$

We than see that if

$$\rho_1' \le \Big\{28\Big[1+\frac{a^*K_0C^*}{2c_*}\Big]^2\Big\}^{-1}\Delta t, \tag{4.25}$$

then

$$\sum_{n=1}^{l-1}|T_L| \le \frac{1}{4}\sum_{n=0}^{l-1}\|d_t\zeta^n\|^2 + C(\Delta t)^4. \tag{4.26}$$

The rest of the proof follows as in the proof of Theorem 4.1.

Similar techniques can be used to give *a priori* error estimates for the approximations given in (2.13) and (2.14) as well as for the corresponding iterative approximations defined in Section 3. Since in the major applications, the coefficients depend upon ∇u (the strain), the techniques presented here will only yield optimal order H^1-estimates instead of the optimal order L^2-estimates obtained in Theorems 4.1 and 4.2.

5. COMPUTATIONAL CONSIDERATIONS

In this section we shall consider some rough operation counts to estimate the computational complexity of the methods presented here. We shall show that the iterative methods presented in Section 3 allow us to obtain near optimal order work estimates. Therefore, these methods are very efficient computationally.

First consider $d = 2$ and the second order hyperbolic equation. Let M be the dimension of $\mathcal{M}_h$ and N be the number of time steps. George[22] has shown in some special cases that the procedure of setting up and factoring L^n (from (3.9)) requires $O(M^{3/2})$ operations and that the solution of (3.2), given the factorization, requires $O(M \log M)$ operations. Hoffman *et al.*[23] have shown that such bounds are minimal. Therefore, if we conjecture the validity of the above estimates for our problem and refactor L^n and solve (3.2) at each time step, the total amount of work done is

$$O(N\{M^{3/2} + M\log M\}) = O(NM^{3/2}). \tag{5.1}$$

We note that the work of factorization dominates the estimates.

Using the preconditioned iterative methods presented in Section 3, one does not have to refactor at every time step. With $d = 2$ we have

$$N \approx (\Delta t)^{-1} \approx h^{-(r/2)} \approx M^{r/4}. \tag{5.2}$$

We are willing to refactor periodically, but our goal is to have the total work estimate (5.1) dominated by the work of solving, $O(NM \log M)$. We shall see that for $r \ge 3$, this goal can be achieved. For $r = 2$ (piecewise linear elements), the work of factoring one matrix is already almost as large as the total work of solving ($O(M^{3/2} \log M)$ in this case). If we refactor and update the preconditioning matrix $N^{1/4}$ equally spaced times, one can show that the preconditioning matrices are sufficiently comparable to the true matrices that each iteration of the iterative procedure yields a norm reduction of $O((\Delta t)^{1/4})$. Then for Δt sufficiently small, five iterations per time step will satisfy the norm reduction requirement of (4.25). Next, for $n = 3$ (piecewise quadratic elements), (5.1) and (5.2) show that the total work is

$$O(N^{1/4}M^{3/2} + 5NM \log M) = O(M^{(3/16)+(3/2)} + M^{7/4} \log M)$$
$$= O(M^{7/4} \log M), \tag{5.3}$$

and the work of solving dominates the estimate. If $r = 2$, the total work is

$$O(M^{(3/2)+(1/8)} + 5M^{3/2} \log M) = O(M^{(13/8)}), \tag{5.4}$$

which is still much better than the $O(M^2)$ work estimate if the matrices are factored at each time step.

If $r \geq 4$, one can refactor and update the preconditioning matrix more frequently (specifically $N^{1/2}$ equally spaced times), obtain a norm reduction of $O((\Delta t)^{1/2})$ with each iteration and by iterating only three times per time step, still have the work of solving dominate the work estimate. If $r = 4$ (piecewise cubics), the total work is

$$O(M^{7/4} + 3M^2 \log M) = O(M^2 \log M). \tag{5.5}$$

We thus see that if $r \geq 3$, then by refactoring and updating the preconditioning matrix sufficiently often (depending upon r), the total work is of the order $O(NM \log M)$. Since the total number of unknowns in the problem is $O(NM)$, we see that we can obtain almost optimal order work estimates for $r \geq 3$.

For $d = 3$, the work of factoring a matrix is $O(M^2)$ while the work of solving the result is $O(M^{4/3})$. Thus if $r \geq 2$ the total work of solving again dominates the work of factoring a matrix. Thus if refactoring is done sufficiently infrequently (depending upon r) the total work of solving will again dominate the total work estimates.

It is computationally wasteful to iterate sufficiently many times at each time step to achieve the pessimistic bounds given by (4.25). Instead, one can monitor the norm reduction actually produced at each step of the iteration and stop iterating when sufficient norm reduction is achieved. Additional stopping criteria can be imposed in this monitoring process. See [14] for a discussion of stopping criteria for a related problem.

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