A GALERKIN PROCEDURE FOR SYSTEMS OF DIFFERENTIAL EQUATIONS (1)

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ABSTRACT - A continuous time and an extrapolated coefficient Crank-Nicolson-Galerkin method are considered for approximating solutions of boundary and initial value problems for a quasi-linear parabolic system of partial differential equations which is coupled to a non-linear system of ordinary differential equations. A priori bounds are derived that reduce the estimation of error to problems in approximation theory. Then approximation theory results yield optimal order rates of convergence for the $H^1(\Omega)$ norm. The extrapolated coefficient method yields linear algebraic equations for strongly non-linear problems.

1. Introduction.

Problems in genetics, nerve impulse transmission, and chemical reactor theory [1, 11] have generated systems of semi-linear parabolic partial differential equations and coupled systems of semi-linear parabolic partial differential equations and non-linear ordinary differential equations. It is clear that the next generation of models will involve some coupled systems of quasi-linear parabolic partial differential equations and non-linear ordinary differential equations. At some point the power of analytic techniques will wane and be replaced with massive computational programs to generate and catalog the stable and unstable solution regions in the data space as well as to observe the type of stable solution that is obtained. Indeed, such computation efforts are probably already underway.

Consider the degenerate system of non-linear initial-boundary value problems

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given by

$$\frac{\partial U^i(x,t)}{\partial t}=\sum_{l=1}^{N_1}\frac{\partial}{\partial x_l}\left(\sum_{j=1}^{N_2}\sum_{m=1}^{N_1}C_{ij}^{lm}\frac{\partial U^j}{\partial x_m}+B_i^l\right)+F_i,\qquad (x,t)\in Q_T,$$

(1.1)

$$\frac{\partial V^k(x,t)}{\partial t}=G_k\qquad\qquad ,\quad (x,t)\in Q_T,$$

for $i=1,2,\dots,N_2$ and $k=1,2,\dots,N_3$ where x lies in a bounded domain$\Omega\subset\mathbf{R}^{N_1}$, $0<t\leq T$, and the solution vectors $U=(U^1,U^2,\dots,U^{N_2})$ and $V=(V^1,V^2,\dots,V^{N_3})$ map $Q_T\equiv\Omega\times(0,T]$ into $\mathbf{R}^{N_2}$ and $\mathbf{R}^{N_3}$ respectively subject to the initial conditions

$$\begin{aligned}U^i(x,0)&=U_0^i(x),\qquad x\in\Omega,\\ V^k(x,0)&=V_0^k(x),\qquad x\in\Omega.\end{aligned}\tag{1.2}$$

The boundary $\partial\Omega$ of Ω and its inner-direct normal $\nu=(\nu_1,\nu_2,\dots,\nu\quad)$ are assumed as smooth as may be needed, and (1.1) is considered subject to the boundary conditions

$$\sum_{l=1}^{N_1}\left(\sum_{j=1}^{N_2}\sum_{m=1}^{N_1}C_{ij}^{lm}\frac{\partial U^j}{\partial x_m}+B_i^l\right)\nu_l=g_i(x,t,U),\ \text{on}\ S_T\equiv\partial\Omega\times(0,T],\tag{1.3}$$

where $g=(g_1,g_2,\dots,g_{N_2})$ is a given map of $S_T\times\mathbf{R}^{N_2}$ into $\mathbf{R}^{N_2}$. We shall assume the following functional dependences for the appropriate indices:

$$C_{ij}^{lm}=C_{ij}^{lm}(x,t,U,V),\ B_i^l=B_i^l(x,t,U,V),\ F_i=$$
$$=F_i(x,t,U,V)\ \text{and}\ G_k=G_k(x,t,U,V).$$

In this paper a continuous Galerkin method and an extrapolated coefficient discrete Crank-Nicolson-Galerkin method for (1.1)-(1.3) are formulated and analyzed. The methods and analysis presented are generalizations of those of Douglas and Dupont [7, 8]. In section two, some notation, definitions and the variational form of (1.1)-(1.3) are noted. Section 3 is devoted to the derivation of the Galerkin methods. An a priori estimate of the error for the continuous Galerkin method is derived in section 4. The analysis of the a priori error for the extrapolated coefficient discrete Crank-Nicolson-Galerkin method is carried out in section 5. The estimates reduce the error estimations to problems in approximation theory. In section 6, some results from approximation theory are presented which, together with the error estimates, determine rates of convergence for a certain class of approximating subspaces.

We emphasize that the extrapolated-coefficient Crank-Nicolson-Galerkin method yields a linear system of equations at each time step even though the underlying problem is quite non-linear.

2. Preliminaries.

The formulation of (1.1)-(1.3) will be condensed through the combined use of summation convention on repeated indices, $\dot{U}$ for $\frac{\partial U}{\partial t}$, $U_{,m}$ for $\frac{\partial U}{\partial x_m}$, and suppression of the arguments. Thus the differential system (1.1) and boundary system (1.3) become

$$
\text{(2.1)}\quad
\begin{aligned}
&a)\quad \dot{U}^i=(C_{ij}^{lm}\,U^j{}_{,m}+B_i^l)_{,l}+F_i, && Q_T,\\
&b)\quad \dot{V}^k=G_k, && Q_T,\\
&c)\quad (C_{ij}^{lm}\,U^j{}_{,m}+B_i^l)\,\nu_l=g_i, && S_T.
\end{aligned}
$$

The Euclidean norm of the appropriate dimension will be denoted $|u|^2=u^i\,u^i$. We next define the inner product

$$
\text{(2.2)}\qquad \langle a, b\rangle\equiv\int_\Omega a\,(x, t)\;b\,(x, t)\;dx.
$$

Summing inner products of components we define

$$
\text{(2.3)}\quad
\begin{aligned}
&\langle \eta, U\rangle_{(1)}\equiv\int_\Omega \eta^i\,(x, t)\;U^i\,(x, t)\;dx, && \text{sum on } i,\\
&\langle \zeta, V\rangle_{(2)}\equiv\int_\Omega \zeta^k\,(x, t)\;V^k\,(x, t)\;dx, && \text{sum on } k,\\
&\langle \eta, U\rangle_{(1),\partial\Omega}\equiv\int_{\partial\Omega} \eta^i\,(x, t)\;U^i\,(x, t)\;ds, && \text{sum on } i,
\end{aligned}
$$

with corresponding norms

$$\|U(\cdot,t)\|^2_{(1),\Omega} \equiv \langle U,U\rangle_{(1)},$$

(2.4) $$\|V(\cdot,t)\|^2_{(2),\Omega} \equiv \langle V,V\rangle_{(2)}, \text{ and}$$

$$\|U(\cdot,t)\|^2_{(1),\partial\Omega} \equiv \langle U,U\rangle_{(1),\partial\Omega}.$$

We next define the norms

$$\|U\|^2_{(1),Q_T} \equiv \int_0^T \int_\Omega U^i U^i \, dx\, dt, \text{ and}$$

(2.5)

$$\|(U)_x\|^2_{(1),Q_T} \equiv \int_0^T \int_\Omega U^i_{,m} U^i_{,m} \, dx\, dt$$

with similar norms indexed (2). We finally define the norm

$$\|(U)_x\|_{Q_T,\infty} = \sup_{(x,t)\in Q_T} \sup_{\substack{j=1,\dots,N_2\\ i=1,\dots,N_1}} \left|\frac{\partial U^j(x,t)}{\partial x_i}\right|, \text{ and}$$

(2.6)

$$\|U\|_{(j),Q,\infty} = \sup_{t\in(0,T]} \|U\|_{(j),\Omega}, \quad j=1,.$$

Let $\{v^\alpha\colon \alpha=1,2,\dots,N_4\}\subset H^1(\Omega)$ denote a basis of a subspace $\mathcal{M}\subset H^1(\Omega)$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with respective norms given by

$$\|\eta\|^2_{\mathcal{H}_1} \equiv \|\eta\|^2_{(1),Q_T} + \|(\eta)_x\|^2_{(1),Q_T} \quad \text{and}$$

(2.7)

$$\|\zeta\|^2_{\mathcal{H}_2} \equiv \|\zeta\|^2_{(2),Q_T}.$$

Let $\mathcal{M}_1\equiv\{\eta\colon \eta^i=\eta^i_\alpha(t)\,v^\alpha(x) \quad i=1,2,\dots,N_2\}\subset\mathcal{H}_1$ and $\mathcal{M}_2\equiv\{\zeta\colon \zeta^k= =\zeta_\alpha{}^k(t)\,v^\alpha(x),\ k=1,2,\dots,N_3\}\subset\mathcal{H}_2$. Multiplying (2.4, a) by $\eta\in\mathcal{M}_1$ and (2.4, b) by $\zeta\in\mathcal{M}_2$ and integrating by parts we obtain

$$i=1,2,\dots,N_2 \text{ and } k=1,2,\dots,N_3,$$

$$
(2.8)\qquad
\begin{aligned}
&\langle \eta^i, \dot{U}^i\rangle = \mathcal{A}^i(t; U, V; U, \eta^i) - \int_{\partial\Omega\times\{t\}} \eta^i g_i \, ds, && \eta \in \mathcal{M}_1,\\
&\langle \zeta^k, \dot{V}^k\rangle = \mathcal{G}^k(t; U, V; \zeta^k), && \zeta \in \mathcal{M}_2,\\
&\langle \eta^i(x,0), U^i(x,0)\rangle = \langle \eta^i(x,0), U_0{}^i\rangle, && \eta(x,0) \in \mathcal{M},\\
&\langle \zeta^k(x,0), V^k(x,0)\rangle = \langle \zeta^k(x,0), V_0{}^k\rangle, && \zeta(x,0) \in \mathcal{M},
\end{aligned}
$$

where here in (2.8) we have not employed the summation convention and where we define for $a, d, e \in \mathcal{H}_1$ and $b, f \in \mathcal{H}_2$,

$$\mathcal{A}^i(t; a, b; d, e) \equiv \mathcal{C}^i(t; a, b; d, e) + \mathcal{B}^i(t; a, b; e) + \mathcal{F}^i(t; a, b; e),$$

$$
(2.9)\qquad
\begin{aligned}
&\mathcal{C}^i(t; a, b; d, e) \equiv -\int_\Omega e^i{}_{,l}\, C_{ij}{}^{lm}(x, t, a, b)\, d^j{}_{,m}\, dx,\\
&\mathcal{B}^i(t; a, b; e) \equiv -\int_\Omega e^i{}_{,l}\, B_i{}^l(x, t, a, b)\, dx,\\
&\mathcal{F}^i(t; a, b; e) \equiv -\int_\Omega e^i F_i(x, t, a, b)\, dx, \quad \text{and}\\
&\mathcal{G}^k(t; a, b; f) \equiv \int_\Omega f^k G_k(x, t, a, b)\, dx,
\end{aligned}
$$

and here in (2.9) we have not summed on i or k. We shall then define additional notation $\mathcal{A}, \mathcal{C}, \mathcal{B}, \mathcal{F}$ and $\mathcal{G}$ to denote $\mathcal{A} = \sum_{i=1}^{N_2} \mathcal{A}^i$, $\mathcal{C} = \sum_{i=1}^{N_2} \mathcal{C}^i$, $\mathcal{B} = \sum_{i=1}^{N_2} \mathcal{B}^i$, $\mathcal{F} = \sum_{i=1}^{N_2} \mathcal{F}^i$ and $\mathcal{G} = \sum_{k=1}^{N_3} \mathcal{G}^k$.

We shall now state our main assumptions. For appropriate indices, $C_{ij}{}^{lm}$ is a continuous function on $\bar{Q}_T \times \mathbf{R}^{N_2} \times \mathbf{R}^{N_3}$ such that

$$(2.10)\qquad \lambda\, |y|^2 \le y_l{}^i\, C_{ij}{}^{lm}\, y^j{}_m \le \mu\, |y|^2, \qquad y \in \mathcal{H}_1,$$

where λ and μ are positive constants. For appropriate indices, $C_{ij}{}^{lm}, B_i{}^l, F_i, G_k$ satisfy the following Lipschitz type behavior

$$|C_{ij}^{lm}(x,t,a,b)-C_{ij}^{lm}(x,t,c,d)|\leq D,$$

$$|B_i^l(x,t,a,b)-B_i^l(x,t,c,d)|\leq D,$$

$$\text{(2.11)}\qquad |F_i(x,t,a,b)-F_i(x,t,c,d)|\leq D,$$

$$|G_k(x,t,a,b)-G_k(x,t,c,d)|\leq D,$$

$$|g_i(x,t,a)-g_i(x,t,c)|\leq K\,|a-c|,$$

where

$$\text{(2.12)}\qquad D=K\,(|a-c|^2+|b-d|^2)^{1/2}.$$

Also, it is no loss of generality to assume that the data functions are smooth and bounded and that solutions of (1.1)-(1.3) are classical with bounded derivatives of any reasonable order.

In the estimates to follow we shall use the following trace inequality (see [8])

$$\text{(2.13)}\qquad ||X||^2_{(1),\partial Q}\leq C\,||X||_{(1),Q}\,||(X)_x||_{(1),Q}\leq$$

$$\leq C^2\,(4\varepsilon)^{-1}\,||X||^2_{(1),Q}+\varepsilon\,||(X)_x||^2_{(1),Q}.$$

3. Galerkin Procedures.

We first define a continuous time Galerkin approximation of the variational problem given in (2.8). We shall require that U and V lie in a finite-dimensional subspace of $H^1(\Omega)\ (=W_2^1(\Omega))$ for each t. We shall approximate U and V of (2.8) by W and Y where

$$W^i=w_\alpha^i(t)\,v^\alpha(x), \qquad i=1,\dots,N_2, \quad \text{and}$$

(3.1)

$$Y^k=y_\alpha^k(t)\,v^\alpha(x), \qquad k=1,\dots,N_3.$$

By replacing U and V in (2.8) by W and Y and noting the separation of variables effect of (3.1) we obtain the following system of ordinary differential equations for the coefficients $w_\alpha^i(t)$ and $y_\alpha^k(t)$, for $i=1,2,\dots,N_2$, $k=1,2,\dots,N_3$, $\alpha=1,2,\dots,N_4$ and $\beta=1,2,\dots,N_4$,

$$
(3.2)\quad
\begin{aligned}
&a)\quad \langle v^\alpha, v^\beta\rangle \dot{w}_\beta{}^i(t) = \mathcal{A}^i(t; W, Y; v^\alpha) - \int_{\partial\Omega\times\{t\}} v^\alpha g_i\, ds, \qquad t\in(0,T],\\
&b)\quad \langle v^\alpha, v^\beta\rangle \dot{y}_\beta{}^k(t) = \mathcal{G}^k(t; W, Y; v^\alpha), \qquad t\in(0,T],\\
&c)\quad \langle v^\alpha, v^\beta\rangle w_\beta{}^i(0) = \langle v^\alpha, U_0{}^i(x)\rangle,\\
&d)\quad \langle v^\alpha, v^\beta\rangle y_\beta{}^k(0) = \langle v^\alpha, V_0{}^k(x)\rangle,
\end{aligned}
$$

where here the repeated index β is summed. Clearly the solution of this coupled system of ordinary differential equations will provide the coefficients of the v^α as in (3.1) to yield the approximate solution W and Y of (2.8).

If we multiply (3.2, a) and (3.2, c) by $\eta_\alpha{}^i(t)$ and then multiply (3.2, b) and (3.2, d) by $\zeta_\alpha{}^i(t)$ and sum on α we obtain for each $i=1,2,\ldots,N_2$ and $k=1,2,\ldots,N_3$,

$$
(3.3)\quad
\begin{aligned}
&\langle \eta^i, \dot{W}^i\rangle = \mathcal{A}^i(t; V, Y; W, \eta) - \int_{\partial\Omega\times\{t\}} \eta^i g_i\, ds,\\
&\langle \zeta^k, \dot{Y}^k\rangle = \mathcal{G}^k(t; W, Y; \zeta),\\
&\langle \eta^i(x,0), W^i(x,0)\rangle = \langle \eta^i(x,0), U_0{}^i\rangle,\\
&\langle \zeta^k(x,0), Y^k(x,0)\rangle = \langle \zeta^k(x,0), V_0{}^k\rangle.
\end{aligned}
$$

This is our continuous time Galerkin approximation to the variational problem.

We now consider an approximation of (3.3) in which the time variable is discretized. Let $t_n = n\Delta t$ where $\Delta t = T/N$ for some positive integer N. Let f_n denote $f(t_n)$. Let W_* and Y_* be approximations of W and Y, respectively, which satisfy

$$
\begin{aligned}
\langle \eta, \frac{W_{*n+1}-W_{*n}}{\Delta t}\rangle = \mathcal{A}&\left(t_{n+1/2}; \frac{W_{*n+1}+W_{*n}}{2}, \frac{Y_{*n+1}+Y_{*n}}{2}; \frac{W_{*n+1}+W_{*n}}{2}, \eta\right)\\
&- \int_{\partial\Omega\times\{t_n+1/2\}} \eta\, g\left(x, t_{n+1/2}, \frac{W_{*n+1}+W_{*n}}{2}\right) ds, \qquad n\geq 0,
\end{aligned}
$$

$$(3.4)\qquad \langle \zeta, \frac{Y_{*n+1}-Y_{*n}}{\Delta t} \rangle = \mathcal{G}\left(t_{n+1/2}; \frac{W_{*n+1}+W_{*n}}{2}, \frac{Y_{*n+1}+Y_{*n}}{2}; \zeta \right), \qquad n \geq 0,$$

$$\langle \eta, W_{*0} \rangle = \langle \eta, U_0 \rangle,$$

$$\langle \zeta, Y_{*0} \rangle = \langle \zeta, V_0 \rangle,$$

for all i, k and α, where the notation has suppressed the dependence on i, k and α, and for all $\eta \in M_3 = \sum_1^{N_3} \oplus \mathcal{M}$ and for all $\zeta \in \mathcal{M}_4 = \sum_1^{N_3} \oplus \mathcal{M}$. We shall call (3.4) our Crank-Nicolson-Galerkin approximation. We note that since $\mathcal{A}$ and $\mathcal{G}$ are non-linear functions, the solution of the difference system (3.4) requires the solution of a non-linear algebraic system at each time step. In order to avoid the need to solve these non-linear systems we introduce the following extrapolated coefficient approximation to (3.4) (see [7, 8, 10]):

$$a)\quad \langle \eta, \frac{W_{*n+1}-W_{*n}}{\Delta t} \rangle = \mathcal{A}\left(t_{n+1/2}; EW_{*n}, EY_{*n}; \frac{W_{*n+1}+W_{*n}}{2}, \eta \right) -$$

$$- \int_{\partial\Omega \times \{t_{n+1/2}\}} \eta\, g\,(x, t_{n+1/2}, E_\partial W_{*n})\, ds, \qquad \eta \in \mathcal{M}_3, \qquad n \geq 0,$$

$$b)\quad \langle \zeta, \frac{Y_{*n+1}-Y_{*n}}{\Delta t} \rangle = \mathcal{G}\,(t_{n+1/2}; EW_{*n}, EY_{*n}; \zeta), \qquad \zeta \in \mathcal{M}_4, \qquad n \geq 0,$$

(3.5)

$$c)\quad \langle \eta, W_{*0} \rangle = \langle \eta, U_0 \rangle, \qquad \eta \in \mathcal{M}_3,$$

$$d)\quad \langle \zeta, Y_{*0} \rangle = \langle \zeta, V_0 \rangle, \qquad \zeta \in \mathcal{M}_4,$$

for all i, k and α where

$$a)\quad EX_n = \frac{3}{2} X_n - \tfrac{1}{2} X_{n-1} \text{ and}$$

(3.6)

$$b)\quad E_\partial X_n = \frac{2X_n - 3X_{n-1} + X_{n-2}}{2}$$

and where again the notation has suppressed the dependencies on i, k and α.

We note that the algebraic problem for (3.5) is linear since it requires the solution of a linear algebraic system at each time step when it is written in a form analogous to (3.2). From (3.6) we see that we must have the values of W_* and Y_* for two previous times in Ω and for three previous times in $\partial\Omega$ to use (3.5) to find W_{*n+1} and Y_{*n+1}. Also we need W_{*i} and Y_{*i} for $i=0, 1, 2$ to get started. A predictor-corrector Crank-Nicolson-Galerkin starting procedure will be discussed in section 5 and 6.

4. A Priori Estimate for the Continuous Time Case.

The error bounds derived in this section will be in terms of a norm of $U-\tilde{\eta}$ and $V-\tilde{\zeta}$ where U and V are solutions to (2.8) and $\tilde{\eta}$ and $\tilde{\zeta}$ are the « best possible » approximations to U and V at a particular time from the space $\mathcal{M}$. The problem of obtaining error bounds is thus reduced to a problem in approximation theory and is discussed further in section 6.

THEOREM 4.1: There exist constants K_1 and K_2 which depend on $T, N_1, N_2, N_3, \lambda, \mu, K, \|(U)_x\|_{Q_T,\infty}$, and $\operatorname{diam}\Omega$ such that, for U and V a solution to (2.8), W and Y, a solution to (3.3), and $\tilde{\eta}$ any function from $\mathcal{M}_1$ and $\tilde{\zeta}$ any function from $\mathcal{M}_2$

$$\|U-W\|^2_{(1),\Omega,\infty}+\|V-Y\|^2_{(2),\Omega,\infty}\le$$

$$\le K_1\left\{\|U-\tilde{\eta}\|^2_{(1),\Omega,\infty}+\|(U-\tilde{\eta})_x\|^2_{(1),Q_T}+\left\|\frac{\partial}{\partial t}(U-\tilde{\eta})\right\|^2_{(1),Q_T}\right.$$
$$\left.+\|V-\tilde{\zeta}\|^2_{(2),\Omega,\infty}+\left\|\frac{\partial}{\partial t}(V-\tilde{\zeta})\right\|^2_{(2),Q_T}\right\} \tag{4.1}$$

and

$$\|(U-W)_x\|^2_{(1),Q_T}\le \tag{4.2}$$

$$\le K_2\left\{\|U-\tilde{\eta}\|^2_{(1),\Omega,\infty}+\|(U-\tilde{\eta})_x\|^2_{(1),Q_T}+\left\|\frac{\partial}{\partial t}(U-\tilde{\eta})\right\|^2_{(1),Q_T}+\right.$$
$$\left.+\|V-\tilde{\zeta}\|^2_{(2),\Omega,\infty}+\left|\frac{\partial}{\partial t}(V-\tilde{\zeta})\right\|^2_{(2),Q_T}\right\}.$$

REMARK: In all of the analysis that follows, the K's will denote positive constants which depend upon only the quantities noted for K_1 and K_2 in the statement of Theorem 4.1.

PROOF: Sum (2.8) and (3.3) on all indices and difference to obtain

$$
\begin{aligned}
&\langle \eta, \dot{U}-\dot{W}\rangle_{(1)}+\langle \zeta, \dot{V}-\dot{Y}\rangle_{(2)}=\\
&=\mathcal{A}(t;U,V;U,\eta)-\mathcal{A}(t;W,Y;W,\eta)+\\
&+\mathcal{G}(t;U,V;\zeta)-\mathcal{G}(t;W,Y;\zeta)-\int_{\partial\Omega}\eta\,[g(x,t,U)-g(x,t,W)]\,ds=\\
&=\mathcal{C}(t;W,Y;U-W,\eta)+\{\mathcal{C}(t;U,V;U,\eta)-\mathcal{C}(t;W,Y;U,\eta)\}+\\
&+\{\mathcal{B}(t;U,V;\eta)-\mathcal{B}(t;W,Y;\eta)\}+\{\mathcal{F}(t;U,V;\eta)-\mathcal{F}(t;W,Y;\eta)\}+\\
&+\{\mathcal{G}(t;U,V;\zeta)-\mathcal{G}(t;W,Y;\zeta)\}-\int_{\partial\Omega}\eta\,[g(x,t,U)-g(x,t,W)]\,ds.
\end{aligned}
\tag{4.3}
$$

We shall integrate (4.3) from 0 to t on τ and then use the following test functions:

$$
\begin{aligned}
&\eta\equiv(U-W)-(\tilde{\eta}-U)=\tilde{\eta}-W\in\mathcal{M}_1, \quad \text{and}\\
&\zeta\equiv(V-Y)+(\tilde{\zeta}-V)=\tilde{\zeta}-Y\in\mathcal{M}_2.
\end{aligned}
\tag{4.4}
$$

We split the terms on the left of (4.3) using (4.4). Integrating the first term on the left by parts we obtain

$$
\begin{aligned}
&\int_0^t\int_\Omega \eta^i(\dot{U}^i-\dot{W}^i)\,dx\,d\tau=\int_0^t\int_\Omega (U^i-W^i)(\dot{U}^i-\dot{W}^i)\,dx\,d\tau+\\
&\quad+\int_0^t\int_\Omega(\tilde{\eta}^i-U^i)(\dot{U}^i-\dot{W}^i)\,dx\,d\tau=\\
&=\tfrac{1}{2}\,\|(U^i-W^i)(\cdot,t)\|^2_\Omega-\tfrac{1}{2}\,\|(U^i-W^i)(\cdot,0)\|^2_\Omega+\int_\Omega(\tilde{\eta}^i-U^i)(U^i-W^i)\Big|_0^t\,dx-\\
&\quad-\int_0^t\int_\Omega(\dot{\tilde{\eta}}^i-\dot{U}^i)(U^i-W^i)\,dx\,d\tau.
\end{aligned}
\tag{4.5}
$$

A similar result holds for the second term on the left side of (4.3). We also split the first term on the right side of (4.3) using (4.4). We see that

$$\text{(4.6)} \qquad \mathcal{C}^i(t; W, Y; U-W, \eta) = \mathcal{C}^i(t; W, U-W, U-W) + \\ - \mathcal{C}^i(t; W, Y; U-W, \tilde{\eta}-U)$$

holds for $i=1, 2, \dots, N_2$. Summing (4.6) on i, using the positive definiteness of C_{ij}^{lm} from (2.10) and the first terms of the splittings described above we obtain from (4.3)

$$\begin{aligned}
&\tfrac{1}{2}\{\|(U-W)(\cdot,t)\|^2_{(1),\Omega} + \|(V-Y)(\cdot,t)\|^2_{(2),\Omega}\} + \lambda\,\|(U-W)_x\|^2_{(1),Q_T} \leq \\
&\leq \tfrac{1}{2}\{\|(U-W)(\cdot,0)\|^2_{(1),\Omega} + \|(V-Y)(\cdot,0)\|^2_{(2),\Omega}\} + \\
&\qquad + \Big[\int_\Omega (\tilde{\eta}-U)^i (U-W)^i\Big|_t^0 dx + \int_\Omega (\tilde{\zeta}-V)^k (V-Y)^k\Big|_t^0 dx + \\
\text{(4.7)}\quad & \qquad + \int_0^t\int_\Omega (\dot{\tilde{\eta}}-\dot{U})^i (U-W)^i\,dx\,d\tau + \int_0^t\int_\Omega (\dot{\tilde{\zeta}}-\dot{V})^k (V-Y)^k\,dx\,d\tau \\
&\qquad + \int_0^t \mathcal{C}(t; W, Y; U-W, \tilde{\eta}-U)\,d\tau + \\
&+ \int_0^t \{\mathcal{C}(t; U, V; U, \eta) - \mathcal{C}(t; W, Y; U, \eta) + \mathcal{B}(t; U, V; \eta) - \mathcal{B}(t; W, Y; \eta) + \\
&+ \mathcal{F}(t; U, V; \eta) - \mathcal{F}(t; W, Y; \eta) + \mathcal{G}(t; U, V; \zeta) - \mathcal{G}(t; W, Y; \zeta) - \\
&- \int_{\partial\Omega} \eta\,(g(x,t,U) - g(x,t,W))\,ds\}\,d\tau.
\end{aligned}$$

We shall use the Schwarz inequality on the terms in $[\cdot]$ in (4.7) and then the trivial inequality

$$\text{(4.8)} \qquad a\,b \leq \varepsilon\,a^2 + (4\varepsilon)^{-1}\,b^2$$

to obtain the necessary bounds. For example,

$$\text{(4.9)}\qquad \left|\int_{\Omega}(\tilde{\eta}-U)^i\,(U-W)^i|_t^0\,dx\right| \le \|(U-\tilde{\eta})\,(\cdot\,,t)\|_{(1),\Omega}\,\|(U-W)\,(\cdot\,,t)\|_{(1),(\Omega)}+$$

$$+\|(U-\tilde{\eta})\,(\cdot\,,0)\|_{(1),\Omega}\,\|(U-W)\,(\cdot\,,0)\|_{(1),(\Omega)}\le$$

$$\le(4\varepsilon_1^{-1})\,\|\,(U-\tilde{\eta})\,(\cdot\,,t)\|^2_{(1),\Omega}+\varepsilon_1\,\|(U-W)\,(\cdot\,,t)\|^2_{(1),(\Omega)}+$$

$$+\tfrac{1}{2}\,\|(U-\tilde{\eta})\,(\cdot\,,0)\|^2_{(1),\Omega}+\tfrac{1}{2}\,\|(U-W)\,(\cdot\,,0)\|^2_{(1),(\Omega)}\le$$

$$\le\varepsilon_1\,\|(U-W)\,(\cdot\,,t)\|^2_{(1),\Omega}+K_3\,\|U-\tilde{\eta}\|^2_{(1),(\Omega),\infty}+\tfrac{1}{2}\,\|(U-W)\,(\cdot\,,0)\|^2_{(1),\Omega}$$

and

$$\text{(4.10)}\qquad \left|\int_0^t\int_{\Omega}(\dot{\tilde{\eta}}-\dot{U})^i\,(U-W)^i\,dx\,d\tau\right|\le\int_0^t\left\|\frac{\partial}{\partial t}\,(U-\tilde{\eta})\right\|_{(1),\Omega}\,\|U-W\|_{(1),\Omega}\,d\tau\le$$

$$\le\int_0^t\left\{\tfrac{1}{2}\,\left\|\frac{\partial}{\partial t}\,(U-\tilde{\eta})\right\|^2_{(1),\Omega}+\tfrac{1}{2}\,\|U-W\|^2_{(1),\Omega}\right\}d\tau\le$$

$$\le\tfrac{1}{2}\left\|\,\frac{\partial}{\partial t}\,(U-\tilde{\eta})\right\|^2_{(1),\Omega}+\tfrac{1}{2}\int_0^t\|U-W\|^2_{(1),\Omega}\,d\tau.$$

We use the above techniques and (2.10) to obtain the following bound:

$$\text{(4.11)}\qquad \left|\int_0^t\mathcal{C}\,(t;\,W,Y;\,U-W,\,\tilde{\eta}-U)\,d\tau\right|=$$

$$=\left|-\int_0^t\int_{\Omega}(\tilde{\eta}^i-U^i)_{,l}\,C_{ij}^{lm}\,(x,\,t,\,W,\,Y)\,(U^j-W^j)_{,m}\,dx\,d\tau\right|\le$$

$$\le\mu\int_0^t\|(\tilde{\eta}-U)_x\|_{(1),\Omega}\,\|(U-W)_x\|_{(1),\Omega}\,d\tau\le$$

$$\le\mu\,\varepsilon_2\,\|(U-W)_x\|^2_{(1),Q_T}+K_4\,(4\varepsilon^{-1}\,\mu)\,\|(\tilde{\eta}-U)_x\|^2_{(1),Q_T}.$$

We next use (2.6), (2.11) and the techniques used above to obtain for the first part of η from (4.4),

$$(4.12)\qquad \left|\int_0^t \{\mathcal{C}(t; U, V; U, U-W) - \mathcal{C}(t; W, Y; U, U-W)\}\, d\tau\right| =$$

$$= \left|-\int_0^t \int_\Omega (U^i - W^i)_{,l}\, \{C_{ij}^{lm}(x, t, U, V) - C_{ij}^{lm}(x, t, W, Y)\}\, U^j_{,m}\, dx\, d\tau\right| \leq$$

$$\leq K\, ||(U)_x||_{Q_T,\infty} \int_0^t \int_\Omega |(U-W)_x|\, (|U-W|^2 + |V-Y|^2)^{1/2}\, dx\, d\tau \leq$$

$$\leq K\, ||(U)_x||_{Q_T,\infty} \int_0^t ||(U-W)_x||_{(1),\Omega}\, \{||U-W||_{(1),\Omega} + ||V-Y||_{(2),\Omega}\}\, d\tau \leq$$

$$\leq 2K\, ||(U)_x||_{Q_T,\infty}\, \varepsilon_2\, ||(U-W)_x||^2_{(1),Q_T} + K\, ||(U)_x||_{Q_T,\infty}\, (4\varepsilon_2)^{-1} \int_0^t \{||U-W||^2_{(1),\Omega} +$$

$$+ ||V-Y||^2_{(2),\Omega}\}\, d\tau,$$

where here K denotes a positive constant depending upon K of (2.12), N_1 and N_2. Similar results hold for the difference in the terms involving $\mathcal{B}$ except that there is no $||(U)_x||_{Q_T,\infty}$ term in those bounds. In a similar way we see that for the other part of η,

$$(4.13)\qquad \left|\int_0^t \{\mathcal{C}(t; U, V; U, \tilde{\eta} - U) - \mathcal{C}(t; W, Y; U, \tilde{\eta} - U)\}\, d\tau\right| \leq$$

$$\leq K\, ||(U)_x||_{Q_T,\infty}\, ||(U-\tilde{\eta})_x||^2_{(1),Q_T} +$$

$$+ \frac{K}{2} ||(U)_x||_{Q_T,\infty} \int_0^t \{||U-W||^2_{(1),\Omega} + ||V-Y||^2_{(2),\Omega}\}\, d\tau.$$

where here K has the same dependence as K in (4.12). Similarly we note that for some K_5 and K_6 we have

$$(4.14)\qquad \left|\int_0^t \{\mathcal{F}(t; U, V; \eta) - \mathcal{F}(t; W, Y; \eta)\}\, d\tau\right| \leq$$

$$\leq K_5 \,||U-\tilde{\eta}||^2_{(1),\Omega,\infty} + K_6 \int_0^t \{||U-W||^2_{(1),\Omega} + ||V-Y||^2_{(2),\Omega}\}\, d\tau.$$

A similar result holds for the $\Delta\, \mathcal{G}$ term with the $||U-\tilde{\eta}||^2_{(1),\Omega,\infty}$ bound replaced by $||V-\tilde{\zeta}||^2_{(2),\Omega,\infty}$. We now use (2.13) and (4.8) to estimate the last part in (4.7) as follows

$$(4.15)\qquad \left|\int_0^t \int_{\partial\Omega} \eta\, [g(x, t, U) - g(x, t, W)]\, ds\, d\tau\right| \leq$$

$$\leq K \int_0^t \{||U-W||_{(1),\partial\Omega} + ||\tilde{\eta} - U||_{(1),\partial\Omega}\}\, ||U-W||_{(1),\partial\Omega}\, d\tau \leq$$

$$\leq K_7 \int_0^t ||U-W||^2_{(1),\Omega}\, d\tau + \varepsilon_2\, ||(U-W)_x||^2_{(1),Q_T} + K_8\, ||U-\tilde{\eta}||^2_{(1),Q_T} +$$

$$+ K_9\, ||(U-\tilde{\eta})_x||^2_{(1),Q_T}.$$

At this point we combine all the results of (4.7) through (4.15) and their counterparts. All terms with ε-multipliers are moved to the left side of (4.7) under the assumption that ε_j, $j=1, 2$, are sufficiently small. We obtain

$$(4.16)\qquad (\tfrac{1}{2} - \varepsilon_1)\{||(U-W)(\cdot, t)||^2_{(1),\Omega} + ||(V-Y)(\cdot, t)||^2_{(2),\Omega}\} +$$

$$+ (\lambda - 2K_{10}\, [||(U)_x||_{Q_T,\infty} + 2 + \mu]\, \varepsilon_2)\, ||(U-W)_x||^2_{(1),Q_T} \leq$$

$$\leq ||(U-W)(\cdot, 0)||^2_{(1),\Omega} + ||(V-Y)(\cdot, 0)||^2_{(2),\Omega} +$$

$$+ K_{11}\, ||U-\tilde{\eta}||^2_{(1),\Omega,\infty} + K_{12}\, ||V-\tilde{\zeta}||^2_{(2),\Omega,\infty} +$$

$$+ \tfrac{1}{2}\left\{\left|\left|\frac{\partial}{\partial t}(U-\tilde{\eta})\right|\right|^2_{(1),Q_T} + \left|\left|\frac{\partial}{\partial t}(V-\tilde{\zeta})\right|\right|^2_{(2),Q_T}\right\} +$$

$$+K_{13}\,(\|(U)_x\|_{Q_T,\infty})\,\|(U-\tilde{\eta})_x\|^2_{(1),Q_T}+$$

$$+K_{14}\int_0^t\{\|(U-W)(\cdot,\tau)\|^2_{(1),\Omega}+\|(V-Y)(\cdot,\tau)\|^2_{(2),\Omega}\}\,d\tau\leq$$

$$\leq K_{15}\,\{\|U-\tilde{\eta}\|^2_{(1),\Omega,\infty}+\|V-\tilde{\zeta}\|^2_{(2),\Omega,\infty}+$$

$$+\left\|\frac{\partial}{\partial t}(U-\tilde{\eta})\right\|^2_{(1),Q_T}+\left\|\frac{\partial}{\partial t}(V-\tilde{\zeta})\right\|^2_{(2),Q_T}+\|(U-\tilde{\eta})_x\|^2_{(1),Q_T}\}+$$

$$+K_{16}\int_0^t\{\|(U-W)(\cdot,\tau)\|^2_{(1),\Omega}+\|(V-Y)(\cdot,\tau)\|^2_{(2),\Omega}\}\,d\tau$$

where we have noted that

$$\text{(4.17)}\qquad \|(U-W)(\cdot,0)\|^2_{(1),\Omega}\leq\|(U-\tilde{\eta})(\cdot,0)\|^2_{(1),\Omega}\leq\|U-\tilde{\eta}\|^2_{(1),\Omega,\infty},\quad\text{and}$$

$$\|(V-Y)(\cdot,0)\|^2_{(2),\Omega}\leq\|(V-\tilde{\zeta})(\cdot,0)\|^2_{(2),\Omega}\leq\|V-\tilde{\zeta}\|^2_{(2),\Omega,\infty}\,.$$

We can now drop the non-negative second term on the left of (4.16), let $\varepsilon_1=1/4$, and use Gronwall's inequality to obtain the bound

$$\text{(4.18)}\qquad \|(U-W)(\cdot,t)\|^2_{(1),\Omega}+\|(V-Y)(\cdot,t)\|^2_{(2),\Omega}\leq$$

$$\leq K_{17}\,e^{4K_{16}T}\Big\{\|U-\tilde{\eta}\|^2_{(1),\Omega,\infty}+\|(U-\tilde{\eta})_x\|^2_{(1),Q_T}+\left\|\frac{\partial}{\partial t}(U-\tilde{\eta})\right\|^2_{(1),Q_T}+$$

$$+\|V-\tilde{\zeta}\|^2_{(2),\Omega,\infty}+\left\|\frac{\partial}{\partial t}(V-\tilde{\zeta})\right\|^2_{(2),Q_T}\Big\}.$$

Then, since the right hand side of (4.18) is independent of t, we obtain the bound (4.1). We can then drop the non-negative first term on the left of (4.16) and use the bound (4.1) in the last term on the right of (4.16) to obtain the bound (4.2).

5. A Priori Estimate for the Extrapolated Coefficient Crank - Nicolson - Galerkin Approximation.

In this section we shall derive a bound for the error induced by using the extrapolated coefficient Crank-Nicolson-Galerkin system (3.5) to approximate the solution of the variational problem (2.8). The bound and techniques are similar to those used in the proof of Theorem 4.1. However, in this case we shall require additional smoothness of U in the variable t since t-derivatives have been replaced by differences. We shall obtain a time discretization of the order of $(\Delta t)^2$.

We first let

$$Z_n = Z(x, t_n) = U_n - W_{*n} \quad \text{and}$$

(5.1)

$$X_n = X(x, t_n) = V_n - Y_{*n}.$$

We now abuse the notation of (5.1) to define for a function f of (x, t) or (x, t_n),

(5.2) $$f_{n+1/2} = \tfrac{1}{2}\,[f(x, t_{n+1}) + f(x, t_n)].$$

We emphasize that $f_{n+1/2}$ is *not* $f(x, t_{n+1/2})$. With this notation we note that from (3.7 b) we have

(5.3) $$E_\partial W_{*n} = 2W_{*n-1/2} - W_{*n-3/2}.$$

We shall also use the notation

(5.4) $$d_t W_{*n} = \frac{W_{*n+1} - W_{*n}}{\Delta t}.$$

Theorem 5.1: Let U and V be a solution of (2.8) for $0 \le t \le T$ and let $W*$ and $Y*$ be a solution of (3.5) for $0 \le t_n \le T$. Let Z_n and X_n be as in (5.1). Suppose that $\dfrac{\partial^3 U^i}{\partial t^2 \partial x_l}, \dfrac{\partial^3 U^i}{\partial t^3}$ and all the second derivatives of U^i for $l = 1, 2, \ldots, N_1$ and $i = 1, 2, \ldots, N_2$ are continuous and bounded by $\tilde{K}$. Then there exists constants $K_{18}, K_{19}, \delta > 0$ and $\tau_0 > 0$ which depend upon $T, N_1, N_2, N_3, \lambda, \mu, K, \tilde{K}, \|(U)_x\|_{Q_T,\infty}$ and diam Ω such that for $\Delta t \le \tau_0$, $\tilde{\eta} \in \mathcal{M}_1$, $\tilde{\zeta} \in \mathcal{M}_2$ and $n = 0, 1, \ldots, N$, we have

(5.5) $$\sup_{0 \le t_n \le T} \{\|Z\|^2_{(1),\Omega} + \|X\|^2_{(2),\Omega}\} + \delta \Delta t \sum_{n=2}^{N-1} \|(Z_{n+1/2})_x\|^2_{(1),\Omega} \le$$

$$\le K_{18} \{\|Z_2\|^2_{(1),\Omega} + \|X_2\|^2_{(1),\Omega} + \Delta t\, [\|(Z_0)_x^2\|_{(1),\Omega} + \|(Z_1)_x\|^2_{(1),\Omega}]\} +$$

$$+ K_{19} \{(\Delta t)^4 + \sup_{0 \quad t \le} [\|\tilde{\eta} - U\|^2_{(1),\Omega} + \|\tilde{\zeta} - V\|^2_{(2),\Omega}] + \Delta t \sum_{n=2}^{N-1} \|((\tilde{\eta} - U)_{n+1/2})_x\|^2_{(1),\Omega} +$$

$$+\Delta t \sum_{n=2}^{N-1}\left[\left\|\frac{(\eta-U)_{n+1/2}-(\eta-U)_{n-1/2}}{\Delta t}\right\|^2_{(1),\Omega}+\left\|\frac{(\zeta-V)_{n+1/2}-(\zeta-V)_{n-1/2}}{\Delta t}\right\|^2_{(2),\Omega}\right]\Bigg\}.$$

PROOF: We first consider the consistency equation which we obtain from (2.8) and (3.5). With the equations summed and the notation of (5.1)-(5.4), we get

$$\langle \eta, d_t U_n+e_n^{(1)}\rangle_{(1)}+\langle \zeta, d_t V_n+e_n^{(2)}\rangle_{(2)}= \tag{5.6}$$

$$=\mathscr{A}(t_{n+1/2};\ EN_n+e_n^{(3)}, EV_n+e_n^{(4)};\ U_{n+1/2}+e_n^{(5)}, \eta)-$$

$$-\int_{\partial\Omega}\eta^i g_i(t_{n+1/2}, E_\partial U_n+e_n^{(6)})\,ds+\mathscr{T}(t_{n+1/2};\ EU_n+e_n^{(7)}, EV_n+e_n^{(8)};\ \zeta),$$

where (see [7, 8])

$$e_n^{(i)}=0\,((\Delta t)^2), \qquad i=1, 2, \dots, 8 \tag{5.7}$$

pointwise and in L^2 and (5.7) holds in H^1 (pointwise) for $i=5$. Subtracting corresponding parts of (3.5) from (5.6) and using (5.1)-(5.4) we obtain the error equation

$$\langle \eta, d_t Z_n+e_n^{(1)}\rangle_{(1)}+\langle \zeta, d_t X_n+e_n^{(2)}\rangle_{(2)}= \tag{5.8}$$

$$=\mathcal{C}(t_{n+1/2};\ EW_{*n}\, EY_{*n};\ Z_{n+1/2}+e_n^{(5)}, \eta)+$$

$$+\{\mathcal{C}(t_{n+1/2};\ EU_n+e_n^{(3)}, EV_n+e_n^{(4)};\ U_{n+1/2}+e^{(5)}, \eta)-$$

$$-\mathcal{C}(t_{n+1/2};\ EW_{*n}, EY_{*n};\ U_{n+1/2}+e_n^{(5)},\eta)\}+$$

$$+\{\mathcal{B}(t_{n+1/2};\ EU_n+e_n^{(3)}, EV_n+e_n^{(4)};\ \eta)-$$

$$-\mathcal{B}(t_{n+1/2};\ EW_{*n}, EY_{*n};\ \eta)\}+$$

$$+\{\mathcal{F}(t_{n+1/2};\ EU_n+e_n^{(3)}, EV_n+e_n^{(4)};\ \eta)-$$

$$-\mathcal{F}(t_{n+1/2};\ EW_{*n}, EY_{*n};\ \eta)\}-$$

$$-\int_{\partial\Omega}\eta^i\,[g_i(t_{n+1/2}, E_\partial U_n+e_n^{(6)})-$$

$$-g_i\,(t_{n+1/2}, E_\partial W_{*n})]\,ds+$$

$$+\{\mathcal{G}\,(t_{n+1/2};\,EU_n+e_n^{(7)}, EV_n+e_n^{(8)};\,\zeta)-$$

$$-\mathcal{G}\,(t_{n+1/2};\,EW_{*n}, EY_{*n};\,\zeta)\}.$$

Equation (5.8) corresponds to (4.3) in the proof of Theorem 4.1. As in that proof we let

$$\text{(5.9)}\quad \begin{aligned} &a)\quad \eta=Z_{n+1/2}+(\tilde{\eta}-U)_{n+1/2}\\ &b)\quad \zeta=X_{n+1/2}+(\tilde{\zeta}-V)_{n+1/2}. \end{aligned}$$

We then split the first three terms of (5.8) using (5.9) to obtain

$$\text{(5.10)}\quad \frac{1}{2\Delta t}\{||Z_{n+1}||^2_{(1),\Omega}-||Z_n||^2_{(1),\Omega}+||X_{n+1}||^2_{(2),\Omega}-||X_n||^2_{(2)\Omega,}\}+\lambda\,||(Z_{n+1/2})_x||^2_{(1),\Omega}\leq$$

$$\leq\langle(U-\tilde{\eta})_{n+1/2}, d_t Z_n\rangle_{(1)}+\langle(V-\tilde{\zeta})_{n+1/2}, d_t X_n\rangle_{(2)}+\langle\eta, e_n^{(1)}\rangle_{(1)}+\langle\zeta, e_n^{(2)}\rangle_{(2)}+$$

$$+\mathcal{C}\,(t_{n+1/2};\,EW_{*n}, EY_{*n};\,Z_{n+1/2}, (\tilde{\eta}-U)_{n+1/2})+\mathcal{C}\,(t_{n+1/2};\,EW_{*n}, EY_{*n};\,e_n^{(5)}, \eta)+$$

$$+\{\mathcal{C}\,(t_{n+1/2};\,EU_n+e_n^{(3)}, EV_n+e_n^{(4)};\,U_{n+1/2}+e_n^{(5)}, \eta)-\mathcal{C}\,(t_{n+1/2};$$

$$EW_{*n}, EY_{*n};\,U_{n+1/2}+e_n^{(5)}, \eta)\}+$$

$$+\{\mathcal{B}\,(t_{n+1/2};\,EU_n+e_n^{(3)}, EV_n+e_n^{(4)};\,\eta)-\mathcal{B}\,(t_{n+1/2};\,EW_{*n}, EY_{*n};)\eta)\}+$$

$$+\{\mathcal{F}\,(t_{n+1/2};\,EU_n+e_n^{(3)}, EV_n+e_n^{(4)};\,\eta)-\mathcal{F}\,(t_{n+1/2};\,EW_{*n}, EY_{*n};\,\eta)\}-$$

$$-\int_{\partial\Omega}\eta^i\,[g_i\,(t_{n+1/2};\,E_\partial U_n+e_n^{(6)})-g_i\,(t_{n+1/2};\,E_\partial W_{*n})]\,ds+$$

$$+\{\mathcal{G}\,(t_{n+1/2};\,EU_n+e_n^{(7)}, EV_n+e_n^{(8)};\,\zeta)-\mathcal{G}\,(t_{n+1/2};\,EW_{*n}, EY_{*n};\,\zeta)\}.$$

We now make straightforward applications of Schwarz' inequality and (4.8) as in the proof of Theorem 4.1. We first split all terms involving $||Z_{n+1}||^2_{(1),\Omega}$, $||X_{n+1}||^2_{(2),\Omega}$ and $||(Z_{n+1/2})_x||^2_{(1),\Omega}$ apart from the others with ε_3 and ε_4 multipliers. We then break all other products of norms apart using (4.8) with $\varepsilon=\frac{1}{2}$. For example, the first term in $\{\cdot\}$ on the right of (5.10) can be estimated as follows:

(5.11)
$$|\mathcal{C}\,(t_{n+1/2};\,EU_n+e_n^{(3)},EV_n+e_n^{(4)};$$
$$U_{n+1/2}+e_n^{(5)},\eta)-\mathcal{C}\,(t_{n+1/2};\,EW_{*n},\;EY_{*n};\,U_{n+1/2}+e_n^{(5)},\eta)|\le$$
$$\le K\{||(U_{n+1/2})_x||_{Q_T,\infty}+||(e_n^{(5)})_x||_{Q_T,\infty}\}\{||EZ_n+e_n^{(3)}||^2_{(1),\Omega}+$$
$$+||EX_n+e_n^{(4)}||^2_{(2),\Omega}\}^{1/2}||(Z_{n+1/2}+\tilde{\eta}-U)_{n+1/2})_x||_{(1),\Omega}\le$$
$$\le\varepsilon_4||(Z_{n+1/2})_x||^2_{(1),\Omega}+K_{20}\{||((\tilde{\eta}-U)_{n+1/2})_x||^2_{(1),\Omega}+$$
$$+(\Delta t)^4+||Z_n||^2_{(1),\Omega}+||Z_{n-1}||^2_{(1),\Omega}+||X_n||^2_{(2),\Omega}+||X_{n-1}||^2_{(2),\Omega}\}.$$

Similarly we use the trace inequality (2.13), (4.8) and (5.3) to obtain

(5.12)
$$\left|\int_{\partial\Omega}\eta^i\,[g_i\,(t_{n+1/2};\,E_\partial\,U_n+e_n^{(6)})-g_i\,(t_{n+1/2}\,E_\partial\,W_{*n})]\;ds\right|\le$$
$$\le K\,||E_\partial\,Z_n+e_n^{(b)}||_{(1),\partial\Omega}\,\{||Z_{n+1/2}||_{(1),\partial\Omega}+||(\tilde{\eta}-U)_{n+1/2}||_{(1),\partial\Omega}\}\le$$
$$\le\varepsilon_4\{||(Z_{n+1/2})_x||^2_{(1),\Omega}+||(Z_{n-1/2})_x||^2_{(1),\Omega}+||(Z_{n-3/2})_x||^2_{(1),\Omega}\}+$$
$$+K_{21}\,(\varepsilon_4^{-1})\,\{(\Delta t)^4+||Z_n||^2_{(1),\Omega}+||Z_{n-1}||^2_{(1),\Omega}+||Z_{n-2}||^2_{(1),\Omega}\}+$$
$$+K_{22}\,(\varepsilon_4^{-1})\,||Z_{n+1}||^2_{(1),\Omega}+K_{23}\{||((\tilde{\eta}-U)_{n+1/2})_x||^2_{(1),\Omega}+||(\tilde{\eta}-U)_{n+1/2}||^2_{(1),\Omega}\}.$$

Combining the above estimates with other corresponding terms, multiplying by $2\Delta t$ and summing from $n=2$ to $n=l-1$ we obtain

(5.13)
$$||Z_l||^2_{(1),\Omega}-||Z_2||^2_{(1),\Omega}+||X_l||^2_{(2),\Omega}-X_2||^2_{(2),\Omega}+2\lambda\Delta t\sum_{n=2}^{l-1}||(Z_{n+1/2})_x||^2_{(1),\Omega}\le$$
$$\le K_{24}\,2\Delta t\,\{||Z_l||^2_{(1),\Omega}+||X_l||^2_{(2),\Omega}\}+K_{25}\,\Delta t\sum_{n=0}^{l-1}\{||Z_n||^2_{(1),\Omega}+||X_n||^2_{(2),\Omega}\}+$$
$$+14\Delta t\,\varepsilon_4\sum_{n=2}^{l-1}||(Z_{n+1/2})_x||^2_{(1),\Omega}+\varepsilon_4\,\Delta t\,\{||(Z_1)_x||^2_{(1),\Omega}+||(Z_0)_x||^2_{(1),\Omega}\}+$$
$$+K_{26}\{(\Delta t)^4+\Delta t\sum_{n=2}^{l-1}[||(\tilde{\eta}-U)_{n+1/2}||^2_{(1),\Omega}+$$
$$||(\tilde{\zeta}-V)_{n+1/2}||^2_{(2),\Omega}+||((\tilde{\eta}-U)_{n+1/2})_x||^2_{(1),\Omega}]\}+$$
$$+2\Delta t\sum_{n=2}^{l-1}\{\langle(U-\tilde{\eta})_{n+1/2},\,d_t\,Z_n\rangle_{(1)}+\langle(V-\tilde{\zeta})_{n+1/2},\,d_t\,X_n\rangle_{(2)}\}.$$

We now sum by parts to bound the first term in the last set of $\{\cdot\}$ in (5.13). We obtain

$$(5.14)\quad \left|2\Delta t \sum_{n=2}^{l-1} \langle (U-\tilde{\eta})_{n+1/2}, d_t Z_n\rangle_{(1)}\right| \leq 2\left|\langle (U-\tilde{\eta})_{l-1/2}, Z_l\rangle_{(1)}\right| +$$

$$+2\left|\langle (U-\tilde{\eta})_{3/2}, Z_2\rangle_{(1)}\right| + \left| 2\Delta t \sum_{n=2}^{l-1} \left\langle \frac{(U-\tilde{\eta})_{n+1/2}-(U-\tilde{\eta})_{n-1/2}}{\Delta t}\right\rangle z_n\right| \leq$$

$$\leq \varepsilon_3 \|Z_l\|^2_{(1),\Omega} + \|Z_2\|^2_{(1),\Omega} + K_{27}\{\|(\tilde{\eta}-U)_{3/2}\|^2_{(1),\Omega} + \|(\eta-U)_{l-1/2}\|^2_{(1),\Omega} +$$

$$+\Delta t \sum_{n=2}^{l-1} \|Z_n\|^2_{(1),\Omega} + \Delta t \sum_{n=2}^{l-1} \left\| \frac{(\tilde{\eta}-U)_{n+1/2}-(\tilde{\eta}-U)_{n-1/2}}{\Delta t}\right\|^2_{(1),\Omega}\Bigg\}.$$

We obtain a similar bound for the last term in the last set of $\{\cdot\}$ in (5.13). We now transport all ε_3 and ε_4 terms to the left of (5.13) and combine our estimates to obtain

$$(5.15)\quad (1-2K_{24}\Delta t-\varepsilon_3)\{\|Z_l\|^2_{(1),\Omega} + \|X_l\|^2_{(2),\Omega}\}+$$

$$+(2\lambda-14\,\varepsilon_4)\,\Delta t \sum_{n=2}^{l-1} \|(Z_{n+1/2})_x\|^2_{(1),\Omega} \leq$$

$$\leq 2\{\|Z_2\|^2_{(1),\Omega} + \|X_2\|^2_{(2),\Omega}\} + \varepsilon_4\,\Delta t\,\{\|(Z_0)_x\|^2_{(1),\Omega} + \|(Z_1)_x\|^2_{(1),\Omega}\}+$$

$$+K_{28}\{(\Delta t)^4 + \sup_{0\leq t\leq T}[\|\tilde{\eta}-U\|^2_{(1),\Omega} + \|\tilde{\zeta}-V\|^2_{(2),\Omega}] + \Delta t\sum_{n=2}^{l-1}\|((\eta-U)_{n+1/2})_x\|^2_{(1),\Omega} +$$

$$+\Delta t \sum_{n=2}^{l-1}\left[\left\|\frac{(\tilde{\eta}-U)_{n+1/2}-(\tilde{\eta}-U)_{n-1/2}}{\Delta t}\right\|^2_{(1),\Omega} + \left\|\frac{(\tilde{\zeta}-V)_{n+1/2}-(\tilde{\zeta}-V)_{n-1/2}}{\Delta t}\right\|^2_{(2),\Omega}\right]\Bigg\} +$$

$$+K_{29}\,\Delta t \sum_{n=0}^{l-1} \{\|Z_n\|^2_{(1),\Omega} + \|X_n\|^2_{(2),\Omega}\}.$$

Choose Δt, ε_3 and ε_4 sufficiently small that the coefficients on the left of (5.15) are positive. Then since (5.15) holds for all l such that $3\leq l\leq N$, we see that an application of the discrete Gronwall lemma gives us the desired result (5.5).

It is clear from (5.5) that the time discretization-error is of the order $(\Delta t)^2$ as desired. The predictor-corrector Crank-Nicolson scheme discussed in the next section will give proper estimates for the first term in $\{\cdot\}$ on the right of (5.5). In the next section we shall use some approximation theory results to determine the rate of convergence from the last terms in (5.5).

6. Determination of Rates of Convergence.

We noted earlier that W_{*i} and Y_{*i} for $i=0, 1, 2$ were needed to start using the extrapolated coefficient Crank-Nicolson-Galerkin method given by (3.5). One method would be to solve the non-linear algebraic equations arising from the Crank-Nicolson-Galerkin scheme given by (3.4). Another method is to use predictor-corrector Crank-Nicolson-Galerkin methods (see [6, 7, 8, 10]. A predictor-corrector and a predictor-corrector-corrector Crank-Nicolson-Galerkin method are presented in [6] for the equation (1.1). In [6] estimates of the form (5.5) without the first bracket on the right are obtained. For this reason the rate of convergence is determined by the terms in the second bracket on the right of (5.5). We shall use some approximation theory results to determine these convergence rates.

Let $||u||_{H^s(\Omega)}$ denotes the s^{th} order Sobolev norm of u on Ω. Let $\{\mathcal{M}_h\}$ be a family of finite-dimensional subspaces of $H^1(\Omega)$ parametrized by h and called an S_h^r family with the following property: for $s=0, 1$ and $s\leq p\leq r$, there exists a constant $\widetilde{K}>0$ such that

$$\inf_{\chi\in\mathcal{M}_h} ||\varphi-\chi||_{H^s(\Omega)} \leq K\, ||\varphi||_{H^p(\Omega)} h^{p-s}, \qquad \varphi\in H^p(\Omega). \tag{6.1}$$

Essentially all the standardly used subspaces satisfy this requirement for some choice of $r\geq 2$. The Hermite spaces, the smooth spline spaces, and the spaces based on triangles are all S_h^r spaces, at least if some modest regularity is practiced in the choice of nodes [2, 3, 4, 5, 8, 9].

We shall use the methods of [7] to yield our final results.

THEOREM 6.1: Let $\mathcal{M}=\mathcal{M}_h$ be a member of an S_h^r family. Let U, V, W and Y be as in Theorem 4.1. Suppose for every $t\in[0, T]$ the components of U and V satisfy

$$||U^i(\cdot, t)||_{H^r(\Omega)} \leq \hat{K}, \qquad i=1, 2, \dots, N_2,$$

$$||V^k(\cdot, t)||_{H^r(\Omega)} \leq \hat{K}, \qquad k=1, 2, \dots, N_3,$$

$$\left\|\frac{\partial}{\partial t} U^i(\cdot, t)\right\|_{H^r(\Omega)} \leq \varphi_i(t), \qquad i=1, 2, \dots, N_2,$$

$$\left\|\frac{\partial}{\partial t} V^k(\cdot, t)\right\|_{H^r(\Omega)} \leq \psi_k(t), \qquad k=1, 2, \dots, N_3,$$

where $\hat{K}$ is independent of t and φ_i and ψ_k are bounded in $L^2(0,T)$. Then there is a constant C such that

$$\|U-W\|_{(1),\Omega,\infty}+\|V-Y\|_{(1),\Omega,\infty}+\|(U-W)_x\|_{(1),Q_T}\leq C\,h^{r-1}. \tag{6.3}$$

We note that the constant C in (6.3) depends upon $\hat{K}$ and the bound for

$$\sup_{i=1,2,\dots,N_2}\int_0^T \varphi_i^2(t)\,dt \quad\text{and}\quad \sup_{k=1,\dots,N_3}\int_0^T \psi_k^2(t)\,dt.$$

PROOF: From (2.3) and (2.4) we note that the norms in (4.1) and (6.3) are just sums of corresponding norms on the components. From (6.2) the desired result follows directly from the corresponding theorems in [7] and (6.1).

THEOREM 6.2: Let $\mathcal{M}=\mathcal{M}_h$ be a member of an S_h^r family. Let U, V, W_*, Y_*, Z and X be as in Theorem 5.1. Let $U, V, \frac{\partial U}{\partial t}$ and $\frac{\partial V}{\partial t}$ satisfy the restrictions of Theorem 6.1. Then there exists a constant C such that

$$\sup_{0\leq t_n\leq T}\{\|(U-W_*)(\cdot,t_n)\|^2_{(1),\Omega}+\|(V-Y_*)(\cdot,t_n)\|^2_{(2),\Omega}\}+ \\ +\Delta t\sum_{n=0}^{N-1}\|(Z_{n+1/2})_x\|^2_{(1),\Omega}\leq C\,h^{2(r-1)}. \tag{6.4}$$

We note that the constant C in (6.4) depends upon $T, N_1, N_2, N_3, \lambda, \mu, K$, $\tilde{K}, \hat{K}, \|(U)_x\|_{Q_T,\infty}$ and a bound for $\sup_{i=1,2,\dots,N_2}\int_0^T \varphi_i^2(t)\,dt$ and $\sup_{k=1,\dots,N_3}\int_0^T \psi_k^2(t)\,dt$.

PROOF: As mentioned at the first of this section, the terms in the first $\{\cdot\}$ on the right of (5.5) can be bounded by a constant times terms in the second $\{\cdot\}$ of (5.5) via a predictor-corrector starter. This observation also allows the index in the sum in (6.4) to start at zero. Then as in [7] we obtain

$$\left\|\frac{(\tilde{\eta}-U)_{n+1/2}-(\tilde{\eta}-U)_{n-1/2}}{\Delta t}\right\|^2_{(1),\Omega}\leq\sum_{i=1}^{N_2}\int_0^T \varphi_i(t)^2\,dt\,h^r. \tag{6.5}$$

Combining this estimate with the method of the proof of Theorem 6.1 and [7] we obtain the desired result.

We note that we have obtained optimal order convergence in the H^1 norm as in [7].

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