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A Coupled Nonlinear Hyperbolic–Parabolic System*

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A boundary initial value problem for a quasi-linear hyperbolic system in one space variable is coupled to a boundary initial value problem for a quasi-linear parabolic equation in two space variables. The coupling occurs through one of the boundary conditions for the hyperbolic system and the source term in the parabolic equation. Such a coupling can arise in the consideration of gas flowing in a porous medium and out of that medium via a pipe. A local existence and uniqueness theorem is demonstrated. The proof involves the method of characteristics, Bernstein's estimates for parabolic partial differential equations, and the contracting mapping theorem.

1. INTRODUCTION

For subsonic fluid flow in a pipe, it is standard practice to use a one-dimensional version of Euler's equations of motion which includes the friction between the fluid and the pipe. By a change of dependent variables this system can be reduced to a standard hyperbolic system [6]. For fluid flow in a porous medium, it is standard practice to use quasi-linear parabolic partial differential equations for the density of the fluid that are derived via Darcy's law [1, 5]. Since pipes are employed to remove fluids from porous media, it seems natural to couple the two models into one system. The standard practice for modelling the removal of fluid from a porous media involves a sink term in the parabolic partial differential equation. Since the sink term is a volumetric flow rate per unit area which equals a linear velocity, it is clear that the fluid velocity at the end of the pipe in the porous medium can be regarded as a volumetric flow rate per unit area and thus is used as part of the sink term in the parabolic partial

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differential equation. The coupling is closed by requiring that the density of the fluid at the end of the pipe in the porous medium be equal to the density of the fluid in the porous medium at the point which represents the location of that end of the pipe.

The preceding considerations motivate the study of the mathematical problem of determining real-valued functions $p = p(z, t)$, $q = q(z, t)$, and $w = w(x, t)$ such that the triple (p, q, w) satisfies

$$\frac{\partial p}{\partial t} + \lambda_1(z, t, p, q) \frac{\partial p}{\partial z} = R_1(z, t, p, q), \qquad 0 < z < 1, \quad 0 < t \leqslant T, \tag{1.1a}$$

$$\frac{\partial q}{\partial t} + \lambda_2(z, t, p, q) \frac{\partial q}{\partial z} = R_2(z, t, p, q), \qquad 0 < z < 1, \quad 0 < t \leqslant T, \tag{1.1b}$$

$$q(1, t) = G(t, p(1, t)), \qquad 0 < t \leqslant T, \tag{1.1c}$$

$$p(z, 0) = p_0(z), \qquad 0 < z < 1, \tag{1.1d}$$

$$q(z, 0) = q_0(z), \qquad 0 < z < 1, \tag{1.1e}$$

$$L(w) = S(x, t, p(0, t), q(0, t)), \qquad (x, t) \in Q_T, \tag{1.1f}$$

$$B(w) = 0, \qquad (x, t) \in S_T, \tag{1.1g}$$

$$w(x, 0) = \varphi(x), \qquad x \in \Omega, \tag{1.1h}$$

$$p(0, t) = \zeta(t, q(0, t), w(0, t)), \qquad 0 < t \leqslant T, \tag{1.1i}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, Ω is an domain in $\mathbb{R}^2$ which contains the origin, $Q_T = \Omega \times \{0 < t \leqslant T\}$, $S_T = \partial\Omega \times \{0 < t \leqslant T\}$, $\partial\Omega$ is the boundary of Ω, L is a parabolic partial differential operator, B denotes a boundary operator, and $\lambda_1, \lambda_2, R_1, R_2, G, p_0, q_0, \varphi, S$, and ζ are known functions of their respective arguments. It is clear that Ω represents the porous medium, $\{0 < z < 1\}$ represents the pipe, and $z = 0$ represents the end of the pipe in the porous medium at a point which is taken to be the origin of the coordinate system for the porous medium.

In what follows, we shall treat two cases of L and B:

Case 1: *Linear.*

$$L(w) \equiv \frac{\partial w}{\partial t} - \sum_{i,j=1}^{2} a_{ij}(x, t) \frac{\partial^2 w}{\partial x_i \, \partial x_j} + \sum_{i=1}^{2} b_i(x, t) \frac{\partial w}{\partial x_i}, \tag{1.2}$$

$$B(w) \equiv \sum_{i,j=1}^{2} a_{ij}(x, t) \frac{\partial w}{\partial x_j} \cos(\mathbf{n}, x_i),$$

where for any $\xi \in \mathbb{R}^2$ of unit length

$$\sum_{i,j=1}^{2} a_{ij}(x, t)\, \xi_i \xi_j \geqslant \nu > 0 \tag{1.3}$$

and $\mathbf{n}$ is the outer normal to S_T.

Case 2: *Quasi-linear.*

$$\begin{aligned} L(w) &\equiv \frac{\partial w}{\partial t} - \sum_{i,j=1}^{2} a_{ij}(x, t, w) \frac{\partial^2 w}{\partial x_i\, \partial x_j} + b(x, t, w, w_x), \\ B(w) &\equiv \sum_{i,j=1}^{2} a_{ij}(x, t, w) \frac{\partial w}{\partial x_j} \cos(\mathbf{n}, x_i\ ,\ ), \end{aligned} \tag{1.4}$$

where for any $\xi \in \mathbb{R}^2$ of unit length

$$0 < \nu \leqslant \sum_{i,j=1}^{2} a_{ij}(x, t, w)\, \xi_i \xi_j \leqslant \mu \tag{1.5}$$

and w_x denotes the gradient of w with respect to x_1 and x_2. We shall make the detailed assumptions upon all coefficients and data as needed below.

Our basic aim is to demonstrate that for T sufficiently small there exists a unique solution of (1.1). Actually we shall carry the analysis for two types of solutions. One type will be called a weak solution and consists essentially of a generalized solution of the hyperbolic part of (1.1) and a classical solution of the parabolic part of (1.1). The other type of solution is the usual classical one. In the next section we formulate the notion of weak solution through a reformulation of the hyperbolic part of (1.1) into integral equations via the characteristics of the hyperbolic equations (1.1a) and (1.1b). We also formulate a mapping $\mathscr{M}$. A fixed point of $\mathscr{M}$ will yield whichever type of solution that the data smoothness will allow. Sections 3, 4, and 5 deal with the preservation of function classes under the mapping $\mathscr{M}$. In particular, Section 3 deals with a priori estimates of the characteristics which are used in Section 4 to obtain the necessary a priori estimates for the solutions of the hyperbolic part of (1.1). We study the a priori estimates of the parabolic cases in Section 5. In Section 6, the results of Sections 3, 4, and 5 are brought together and the preservation of some function classes under the mapping $\mathscr{M}$ is demonstrated. The remainder of Section 6 is devoted to demonstrating that $\mathscr{M}$ is continuous and a contraction. The paper is concluded with a statement summarizing the analysis and results.

2. A Weaker Formulation of (1.1)

Since (1.1) arises from physical considerations, it is natural to assume that the λ_i and R_i, $i = 1, 2$, are smooth bounded functions that are defined on θ_T

$\{(z, t, p, q): 0 \leqslant z \leqslant 1, 0 \leqslant t \leqslant T, -\infty < p < \infty, -\infty < q < \infty\}$. Also, it is natural to assume that there exists a constant $\delta > 0$ such that in θ_T,

$$\lambda_2 < -\delta < 0 < \delta < \lambda_1 . \tag{2.1}$$

For a classical smooth solution (p, q, w) of (1.1), we can define the characteristics [2, 3, 10]

$$z_i = z_i(\tau; z, t), \qquad \max(0, t_i) \leqslant \tau \leqslant t, \qquad i = 1, 2, \tag{2.2}$$

as solutions of the initial value problems

$$dz_i/d\tau = \lambda_i(z_i, \tau, p(z_i, \tau), \quad q(z_i, \tau)), \quad \max(0, t_i) \leqslant \tau \leqslant t, \quad i = 1, 2, \tag{2.3a}$$

$$z_i(t) = z, \qquad i = 1, 2, \tag{2.3b}$$

where

$$t_i = t_i(z, t), \qquad i = 1, 2, \tag{2.4}$$

denotes the unique time at which the characteristic z_i assumes the value $z = i - 1$. Here, we have extended the functions p and q to negative τ via the initial conditions p_0 and q_0. Given the boundedness of the λ_i, we can take T sufficiently small that no more than one of t_1 and t_2 is positive for any (z, t) such that $0 \leqslant z \leqslant 1$ and $0 \leqslant t \leqslant T$. In other words, we can restrict T so that we have at most one bounce of a characteristic to handle.

Integrating (1.1a) and (1.1b) along their respective characteristics [2, 3, 10], we see that for $0 \leqslant t \leqslant T$ any classical solution (p, q, w) of (1.1) must satisfy

(H) *the hyperbolic part of* (1.1):

$$p(z, t) = p_0(z_1(0; z, t)) + \int_0^t R_1(z_1(\tau; z, t), \tau, p(z_1(\tau; z, t), \tau), q(z_1(\tau; z, t), \tau))\, d\tau, \tag{2.5a}$$

$$q(z, t) = q_0(z_2(0; z, t)) + \int_0^t R_2(z_2(\tau; z, t), \tau, p(z_2(\tau; z, t), \tau), q(z_2(\tau; z, t), \tau))\, dt, \tag{2.5b}$$

or (2.5b) and

$$\begin{aligned} p(z, t) &= \zeta(t_1(z, t), q(0, t_1(z, t)), w(0, t_1(z, t))) \\ &\quad + \int_{t_1(z,t)}^t R_1(z_1(\tau; z, t), \tau, p(z_1(\tau; z, t), \tau), q(z_1(\tau; z, t), \tau))\, d\tau, \end{aligned} \tag{2.6}$$

where $q(0, t_1(z, t))$ is computed by replacing z and t in (2.5b) by 0 and $t_1(z, t)$, respectively, or (2.5a) and

$$\begin{aligned} q(z, t) &= G(t_2(z, t), p(1, t_2(z, t))) \\ &\quad + \int_{t_2(z,t)}^t R_2(z_2(\tau; z, t), \tau, p(z_2(\tau; z, t), \tau), q(z_2(\tau; z, t), \tau))\, d\tau, \end{aligned} \tag{2.7}$$

where $p(1, t_2(z, t))$ is computed from (2.5a) via the replacement of z and t by 1 and $t_2(z, t)$, respectively; and

(P) *the parabolic part of* (1.1):

$$\begin{aligned} L(w) &= S(x, t, p(0, t), q(0, t)), && (x, t) \in Q_T, \\ B(w) &= 0, && (x, t) \in S_T, \\ w(x, 0) &= \varphi(x), && x \in \Omega. \end{aligned} \tag{2.8}$$

DEFINITION. A weak solution of (1.1) is any triple of functions (p, q, w) such that p and q are continuous for $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant T$, w is a classical solution of (P) the parabolic part of (1.1) that is described by (2.8), where L and B are discussed in (1.2) and (1.4), and p and q satisfy (H) the hyperbolic part of (1.1) that is described by (2.5a), (2.5b), (2.6), and (2.7).

As indicated above, our solution technique involves the contracting mapping theorem. Actually, the mapping involved here is easy to define. We take $w(0, t)$ in (2.6) and replace it by a function $v = v(t)$. After solving the hyperbolic part of (1.1) for p and q, we substitute $p(0, t)$ and $q(0, t)$ into (1.1f) and solve the parabolic part of (1.1) for w. The mapping $\mathscr{M}$ is obtained by setting

$$w(0, t) = \mathscr{M}v(t). \tag{2.9}$$

Before we can demonstrate that $\mathscr{M}$ is a contraction for T sufficiently small, it is necessary to obtain a few estimates.

3. A PRIORI ESTIMATES ON THE SOLUTIONS OF THE CHARACTERISTIC EQUATIONS

We first recall a lemma from the theory of ordinary differential equations.

LEMMA 3.1 [8]. *Let y and Y be two functions satisfying*

$$y' = f(x, y), \qquad |x - a| \leqslant h, \tag{3.1a}$$

$$y(a) = \alpha, \tag{3.1b}$$

$$Y' = F(x, Y), \qquad |x - a| \leqslant h, \tag{3.1c}$$

$$Y(a) = \beta. \tag{3.1d}$$

Then, for $|x - a| \leqslant h$,

$$|Y(x) - y(x)| \leqslant \exp\{Mh\}[|\alpha - \beta| + \sup |f - F|], \tag{3.2}$$

where h is a positive constant and M is the maximum of the uniform Lipschitz constants on f and F.

We begin with the assumption that $p = p(z, t)$ and $q = q(z, t)$ are uniformly Lipschitz continuous in z and t with Lipschitz constant $K > 1$. Also we assume that their first derivatives are uniformly Hölder continuous with Hölder constant $\tilde{K} > 1$ and with an exponent α, $0 < \alpha < 1$, which will be determined by the parabolic part of (1.1). Next, we assume that λ_1 and λ_2 possess Lipschitz continuous first partial derivatives in θ_T (and we let C denote a positive constant which bounds them in absolute value), all of their first derivatives, and the Lipschitz constants of their first derivatives. We can suppose that $C > 1$.

From the mean value theorem, it follows that

$$| \lambda_i(z^*, \tau, p(z^*, \tau), q(z^*, \tau)) - \lambda_i(z_*, \tau, p(z_*, \tau), q(z_*, \tau))| \leqslant 3CK \,| z^* - z_* |, \qquad i = 1, 2, \tag{3.3}$$

and that a similar estimate holds for the τ variable. Consequently, we employ $3CK$ as the Lipschitz constant indicated in Lemma 3.1. Recalling (2.4), we let

$$t_i^{(j)} = t_i(z^{(j)}, t), \qquad j = 1, 2, \tag{3.4}$$

denote the times that the characteristics z_i, $i = 1, 2$, emanating from the points $(z^{(j)}, t)$ strike the boundary $z = i - 1$.

LEMMA 3.2. *For* $\max\{0, t_i^{(1)}, t_i^{(2)}\} \leqslant \tau \leqslant t \leqslant T$,

$$| z_i(\tau; z^{(1)}, t) - z_i(\tau; z^{(2)}, t)| \leqslant \exp\{3CKt\} \,| z^{(1)} - z^{(2)} |. \tag{3.5}$$

Proof. The result follows from an elementary application of Lemma 3.1 and is therefore omitted.

Considering Eqs. (2.3a) and (2.3b), we obtain by integration

$$z^{(j)} = (i - 1) + \int_{t_i^{(j)}}^{t} \lambda_i(z_i(\tau; z^{(j)}, t), \tau, p(z_i(\tau; z^{(j)}, t), \tau), q(z_i(\tau; z^{(j)}, t), \tau))\, d\tau, \qquad j = 1, 2, \qquad i = 1, 2. \tag{3.6}$$

Subtracting $j = 2$ of (3.6) from $j = 1$, we obtain

$$z^{(1)} - z^{(2)} = \int_{\max(t_i^{(1)}, t_i^{(2)})}^{t} \{\lambda_i^{(1)} - \lambda_i^{(2)}\}\, d\tau \pm \int_{\min(t_i^{(1)}, t_i^{(2)})}^{\max(t_i^{(1)}, t_i^{(2)})} \lambda_i \, d\tau, \tag{3.7}$$

where the choice of sign and argument for the second term depends upon the $t_i^{(j)}$, $j = 1, 2$. Solving for the second term on the right-hand side of (3.7), recalling (2.1) and using (3.3) and Lemma 3.2, we see that the following result is valid.

LEMMA 3.3. *For* $i = 1, 2$,

$$| t_i(z^{(1)}, t) - t_i(z^{(2)}, t)| \leqslant \delta^{-1}(1 + 3CKt \exp\{3CKt\}) \cdot | z^{(1)} - z^{(2)} |. \tag{3.8}$$

Integrating along the characteristics emanating from $(z, t^{(1)})$ and $(z, t^{(2)})$, we employ an argument similar to that of (3.6) and (3.7) and an application of Gronwall's lemma to obtain the following estimate.

LEMMA 3.4. *For* $0 \leqslant t^{(j)} \leqslant T, j = 1, 2$, *and*

$$\max(0, t_i^{(1)}, t_i^{(2)}) \leqslant \tau \leqslant \min(t^{(1)}, t^{(2)}),$$

$$|z_i(\tau; z, t^{(1)}) - z_i(\tau; z, t^{(2)})| \leqslant C \exp\{3CKT\} \mid t^{(1)} - t^{(2)} \mid, \tag{3.9}$$

where we have used $t_i^{(j)}$ *to denote the time that the characteristic* z_i *emanating from* $(z, t^{(j)})$ *strikes the boundary* $z = i - 1$.

An argument similar to those of the two preceding results yields our last Lipschitz estimate.

LEMMA 3.5. *For* $0 \leqslant t^{(j)} \leqslant T, j = 1, 2$,

$$| t_i(z, t^{(1)}) - t_i(z, t^{(2)})| \leqslant \delta^{-1}C(1 + 3CKT \exp\{3CKT\}) \cdot | t^{(1)} - t^{(2)} |. \tag{3.10}$$

We now consider estimates of the first derivatives of the z_i with respect to z and t. Differentiating (2.3a) and (2.3b), we obtain

$$\frac{d}{d\tau}\left(\frac{\partial z_i}{\partial z}\right) = \left\{\frac{\partial \lambda_i}{\partial z} + \frac{\partial \lambda_i}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial \lambda_i}{\partial q}\frac{\partial q}{\partial z}\right\}\frac{\partial z_i}{\partial z}, \qquad \frac{\partial z_i}{\partial z}(t; z, t) = 1, \tag{3.11}$$

and

$$\begin{gathered}\frac{d}{d\tau}\left(\frac{\partial z_i}{\partial t}\right) = \left\{\frac{\partial \lambda_i}{\partial z} + \frac{\partial \lambda_i}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial \lambda_i}{\partial q}\frac{\partial q}{\partial z}\right\}\frac{\partial z_i}{\partial t},\\ \frac{\partial z_i}{\partial t}(t; z, t) = -\lambda_i(z, t, p(z, t), q(z, t)).\end{gathered} \tag{3.12}$$

The solutions of these equations are represented by

$$\frac{\partial z_i}{\partial z} = \exp\left\{\int_t^\tau \left\{\frac{\partial \lambda_i}{\partial z} + \frac{\partial \lambda_i}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial \lambda_i}{\partial q}\frac{\partial q}{\partial z}\right\} d\eta\right\} \tag{3.13}$$

and

$$\frac{\partial z_i}{\partial t} = -\lambda_i(z, t, p(z, t), q(x, t))\frac{\partial z_i}{\partial z}, \tag{3.14}$$

where the functions in the integral of (3.13) are evaluated along the ith characteristic emanating from (z, t). Recalling (3.6) and removing the index (j) from it, it follows from Leibnitz's rule that

$$\begin{aligned}\frac{\partial t_i}{\partial z} &= \{\lambda_i(i-1, t_i, p(i-1, t_i), q(i-1, t_i))\}^{-1}\\ &\quad\cdot \left\{\int_{t_i}^t \left\{\frac{\partial \lambda_i}{\partial z} + \frac{\partial \lambda_i}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial \lambda_i}{\partial q}\frac{\partial q}{\partial z}\right\}\frac{\partial z_i}{\partial z}\, d\tau - 1\right\}\\ &= \{\lambda_i(i-1, t_i, p(i-1, t_i), q(i-1, t_i))\}^{-1} \cdot \left\{-\frac{\partial z_i}{\partial z}(t_i; z, t)\right\}\end{aligned} \tag{3.15}$$

and that

$$\frac{\partial t_i}{\partial t} = \{\lambda_i(i-1, t_i, p(i-1, t_i), q(i-1, t_i))\}^{-1} \cdot \Big\{\lambda_i(z, t, p(z, t), q(z, t)) + \int_{t_i}^{t} \Big\{\frac{\partial \lambda_i}{\partial z} + \frac{\partial \lambda_i}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial \lambda_i}{\partial q}\frac{\partial q}{\partial z}\Big\} \frac{\partial z_i}{\partial t}\, d\tau\Big\} = -\{\lambda_i(i-1, t_i, p(i-1, t_i), q(i-1, t_i))\}^{-1} \frac{\partial z_i}{\partial t}(t_i; z, t), \tag{3.16}$$

where the integrands in (3.15) and (3.16) are evaluated along the ith characteristic emanating from (z, t). Since C bounds λ_i and its derivatives and K denotes the Lipschitz constant for p and q, it follows from simple estimations of (3.13), (3.14), (3.15), and (3.16) that for $0 \leqslant z \leqslant 1$ and $\max(0, t_i) \leqslant \tau \leqslant t \leqslant T$,

$$\left|\frac{\partial z_i}{\partial z}\right| \leqslant \exp\{3CKt\}, \qquad i = 1, 2, \tag{3.17a}$$

$$\left|\frac{\partial z_i}{\partial t}\right| \leqslant C \exp\{3CKt\}, \qquad i = 1, 2, \tag{3.17b}$$

$$\left|\frac{\partial t_i}{\partial z}\right| \leqslant \delta^{-1} \exp\{3CKt\}, \qquad i = 1, 2, \tag{3.17c}$$

$$\left|\frac{\partial t_i}{\partial t}\right| \leqslant \delta^{-1} C \exp\{3CKt\}, \qquad i = 1, 2. \tag{3.17d}$$

Utilizing the representations, bounds, and preceding lemmas, the following estimates can be derived from a liberal use of the Mean Value Theorem and the triangle inequality.

LEMMA 3.6. *Let* $0 \leqslant t^{(j)} \leqslant T \leqslant 1$ *and* $0 \leqslant z^{(j)} \leqslant 1$, $j = 1, 2$. *For* $\max(0, t_i^{(1)}, t_i^{(2)}) \leqslant \tau \leqslant t \leqslant T$,

$$\left|\frac{\partial z_i}{\partial z}(\tau; z^{(1)}, t) - \frac{\partial z_i}{\partial z}(\tau; z^{(2)}, t)\right| \leqslant C_1 \,| z^{(1)} - z^{(2)} |^{\alpha}, \qquad i = 1, 2, \tag{3.18a}$$

$$\left|\frac{\partial z_i}{\partial t}(\tau; z^{(1)}, t) - \frac{\partial z_i}{\partial t}(\tau; z^{(2)}, t)\right| \leqslant C_1 \,| z^{(1)} - z^{(2)} |^{\alpha}, \qquad i = 1, 2, \tag{3.18b}$$

for $\max(t_i^{(1)}, t_i^{(2)}) \leqslant \tau \leqslant \min(t^{(1)}, t^{(2)})$,

$$\left|\frac{\partial z_i}{\partial z}(\tau; z, t^{(1)}) - \frac{\partial z_i}{\partial z}(\tau; z, t^{(2)})\right| \leqslant C_1 \,| t^{(2)} - t^{(2)} |^{\alpha}, \qquad i = 1, 2, \tag{3.19a}$$

$$\left|\frac{\partial z_i}{\partial t}(\tau; z, t^{(1)}) - \frac{\partial z_i}{\partial t}(\tau; z, t^{(2)})\right| \leqslant C_1 \,| t^{(1)} - t^{(2)} |^{\alpha}, \qquad i = 1, 2, \tag{3.19b}$$

and

$$\left| \frac{\partial t_i}{\partial z}(z^{(1)}, t^{(1)}) - \frac{\partial t_i}{\partial z}(z^{(2)}, t^{(2)}) \right| \leqslant C_1\{| z^{(1)} - z^{(2)} |^{\lambda} + | t^{(1)} - t^{(2)} |^{\lambda}\},$$
$$i = 1, 2, \qquad (3.20a)$$

$$\left| \frac{\partial t_i}{\partial t}(z^{(1)}, t^{(1)}) - \frac{\partial t_i}{\partial t}(z^{(2)}, t^{(2)}) \right| \leqslant C_1\{| z^{(1)} - z^{(2)} |^{\lambda} + | t^{(1)} - t^{(2)} |^{\lambda}\},$$
$$i = 1, 2, \qquad (3.20b)$$

where the restriction

$$T < (3KC)^{-1} \qquad (3.21)$$

has been employed to simplify

$$C_1 = 300C^4K[1 + \tilde{K}t]. \qquad (3.22)$$

Remark. For arguments similar to the ones indicated but omitted above, see [2, 3, 10].

4. A Priori Estimates for the Solution of the Hyperbolic Part of (1.1)

Recall from Section 2 that the hyperbolic part of (1.1) consists of the integral equations (2.5a), (2.5b), (2.6), and (2.7) which involve the solutions $z_i(\tau; z, t)$, $i = 1, 2$, of the characteristic equations (2.3a) and (2.3b). Consequently, the results of Section 3 will be used heavily in the estimates that are derived below for $p = p(z, t)$ and $q = q(z, t)$. Before we begin the discussion of the estimates, it is necessary to state some assumptions upon the data functions R_1, R_2, ζ, G, p_0, and q_0.

We assume that R_1 and R_2 possess Lipschitz continuous first derivatives in θ_T. Next, we assume that $\zeta = \zeta(t, q, w)$ possesses Lipschitz continuous first derivatives on $\{(t, q, w): 0 \leqslant t \leqslant T, -\infty < q < \infty, -\infty < w < \infty\}$ and that $G = G(t, p)$ possesses Lipschitz continuous derivatives on $\{(t, p): 0 \leqslant t \leqslant T, -\infty < p < \infty\}$. With respect to the initial conditions p_0 and q_0, we assume that they possess a Lipschitz continuous derivative for $0 \leqslant z \leqslant 1$. It is economical to assume that C used in Section 3 also provides here a bound in absolute value for all of the above functions, all of their first derivatives, and for all of the Lipschitz constants. Since our stated purpose is to obtain a solution of (1.1) which involves at least continuous p and q, it is necessary that our data satisfy the compatibility conditions.

$$p_0(0) = \zeta(0, q_0(0), \varphi(0)), \qquad (4.1a)$$

$$q_0(1) = G(0, p_0(1)). \qquad (4.1b)$$

All of the estimates in this section are aimed at showing that the mapping $\mathcal{M}$ defined in Section 2 by (2.9) will preserve the function classes that contain the solution of (1.1). As is Section 3, we shall assume that $p = p(z, t)$ and $q = q(z, t)$ are uniformly Lipschitz continuous in z and t with Lipschitz constant K and that they possess Hölder continuous first partial derivatives with Hölder constant $\tilde{K}$ and with an exponent α. In obtaining the mapping, we replaced $w(0, t)$ in (2.6) by a function $v = v(t)$. We shall assume that, for $0 \leqslant t \leqslant T$, v possesses a bounded first derivative which is Hölder continuous with exponent α. We shall let $\tilde{C}$, V, and $\tilde{V}$ denote, respectively, the bound in absolute value for v, the Lipschitz constant for v, and the Hölder constant for the derivative of v.

The estimates obtained below may seem strange in that we obtain an estimate for the Lipschitz constant for p and q after having assumed one. The reader should bear in mind that the mapping $\mathcal{M}$ of Section 2 requires that we solve an auxiliary hyperbolic system. This requires a mapping of its own (see, for example, [2, 3, 10]). Consequently, our estimates must reflect the preservation of the necessary function classes through that mapping as well.

An easy estimation of (2.5a), (2.5b), (2.6), and (2.7) provides us with our first estimate.

LEMMA 4.1. *For* $0 \leqslant z \leqslant 1$ *and* $0 \leqslant t \leqslant T$,

$$|p(z, t)| \leqslant (1 + t)C, \tag{4.2a}$$

$$|q(z, t)| \leqslant (1 + t)C. \tag{4.2b}$$

Next, we see from (3.17) and the restriction (3.21), $3CKt \leqslant 1$, that

$$C_2 = 3\delta^{-1}C \tag{4.3}$$

can be used as a uniform Lipschitz constant for z_i and t_i. Without loss of generality we can assume that $C_2 > 1$.

In order to estimate the Lipschitz constants for p and q from (2.5a), (2.5b), (2.6), and (2.7), we must consider three basic cases for characteristics $z_i^{(j)}$, $j = 1, 2$, emanating, respectively, from $(z^{(1)}, t^{(1)})$ and $(z^{(2)}, t^{(2)})$.

Case I. We assume that neither characteristic $z_i^{(j)}$, $j = 1, 2$, hits a lateral boundary before hitting the base $t = 0$.

Case II. We assume that both characteristics $z_i^{(j)}$, $j = 1, 2$, hit the same lateral boundary prior to $t = 0$.

Case III. We assume that only one of the characteristics $z_i^{(j)}$, $j = 1, 2$, hits a lateral boundary while the other hits the base $t = 0$.

In all, there are 5 cases to be considered, but it is no loss of generality to restrict our attention to the lateral boundary $z = 0$ and the behavior of $z_i^{(j)}$, $j = 1, 2$. In fact, the analysis for the boundary $z = 1$ follows the same pattern with the omission of the effect of the function $v = v(t)$.

For Case I, we obtain the following result.

LEMMA 4.2. *When neither characteristic hits the lateral boundary prior to* $t = 0$ *and when* $0 \leqslant t^{(j)} \leqslant t, 0 \leqslant z^{(j)} \leqslant 1, j = 1, 2,$

$$| p(z^{(1)}, t^{(1)}) - p(z^{(2)}, t^{(2)})| \leqslant C_2(2C + 3CKt)\{| z^{(1)} - z^{(2)} | + | t^{(1)} - t^{(2)} |\}, \tag{4.4a}$$

$$| q(z^{(1)}, t^{(1)}) - q(z^{(2)}, t^{(2)})| \leqslant C_2(2C + 3CKt)\{| z^{(1)} - z^{(2)} | + | t^{(1)} - t^{(2)} |\}. \tag{4.4b}$$

Proof. It suffices to consider z_1 and (2.5a) since a similar argument will hold for z_2 and (2.5b). Substituting $(z^{(j)}, t^{(j)})$, $j = 1, 2$, into (2.5a) and differencing them we obtain

$$\begin{aligned} p(z^{(1)}, t^{(1)}) &- p(z^{(2)}, t^{(2)}) \\ &= p_0(z_1(0; z^{(1)}, t^{(1)})) - p_0(z_1(0; z^{(2)}, t^{(2)})) \\ &+ \int_0^{\min(t^{(1)}, t^{(2)})} \{R_1(z_1(\tau; z^{(1)}, t^{(1)}), \tau, p(z_1(\tau; z^{(1)}, t^{(1)}), \tau), q(z_1(\tau; z^{(1)}, t^{(1)}), \tau)) \\ &- R_1(z_1(\tau; z^{(2)}, t^{(2)}), \tau, p(z_1(\tau; z^{(2)}, t^{(2)}), \tau), q(z_1(\tau; z^{(2)}, t^{(2)}), \tau))\} \, d\tau \\ &\pm \int_{\min(t^{(1)}, t^{(2)})}^{\max(t^{(1)}, t^{(2)})} R_1 \, d\tau. \end{aligned} \tag{4.5}$$

Employing the mean value theorem, Lemmas 3.2, and 3.4, we see that

$$\begin{aligned} &| p(z^{(1)}, t^{(1)}) - p(z^{(2)}, t^{(2)})| \\ &\quad \leqslant \{C_2 C + 3CKC_2 \min(t^{(1)}, t^{(2)})\}\{| z^{(1)} - z^{(2)}| + | t^{(1)} - t^{(2)}|\} + C \,| t^{(1)} - t^{(2)}| \end{aligned} \tag{4.6}$$

from which (4.4a) follows.

For Case II, we see from (2.6) that the difference in $p(z^{(1)}, t^{(1)})$ and $p(z^{(2)}, t^{(2)})$ would involve a difference in the ζ terms, an integral of the form similar to that of the first integral on the right side of (4.5), and two integrals similar to the form of the second integral on the right of (4.5) with one of them involving the limits $t_1(z^{(1)}, t^{(1)})$ and $t_1(z^{(2)}, t^{(2)})$. An application of the mean value theorem and repeated use of the Lemmas 3.2 through 3.5 yields the following result.

LEMMA 4.3. *When both characteristics hit a lateral boundary prior to* $t = 0$ *and when* $0 \leqslant z^{(j)} \leqslant 1, 0 \leqslant t^{(j)} \leqslant t, j = 1, 2,$

$$\begin{aligned} &| p(z^{(1)}, t^{(1)}) - p(z^{(2)}, t^{(2)})| \\ &\qquad \leqslant CC_2\{5C + 5CKt + V\} \cdot \{| z^{(1)} - z^{(2)}| + | t^{(1)} - t^{(2)}|\}, \end{aligned} \tag{4.7a}$$

$$\begin{aligned} &| q(z^{(1)}, t^{(1)}) - q(z^{(2)}, t^{(2)})| \\ &\qquad \leqslant CC_2\{5C + 5CKt\} \cdot \{| z^{(1)} - z^{(2)}| + | t^{(1)} - t^{(2)}|\}. \end{aligned} \tag{4.7b}$$

Consider now Case III. We know from the theory of ordinary differential equations [8] that there is a unique characteristic z_1 with characteristic direction λ_1 passing through the origin. Uniqueness also assures us that this characteristic does not intersect $z_1(\tau; z^{(1)}, t^{(1)})$ or $z_1(\tau; z^{(2)}, t^{(2)})$. From the Jordan Curve Theorem and elementary considerations we see that this characteristic must intersect the line segment connecting $(z^{(1)}, t^{(1)})$ and $(z^{(2)}, t^{(2)})$ at a point which we denote as (z^*, t^*). We may thus denote this characteristic by $z_1(\tau; z^*, t^*)$. Assuming without loss of generality that $z_1(\tau; z^{(1)}, t^{(1)})$ is the characteristic that hits $z = 0$ prior to $t = 0$, we may apply the triangle inequality to obtain

$$\begin{aligned} &| p(z^{(1)}, t^{(1)}) - p(z^{(2)}, t^{(2)})| \\ &\quad \leqslant | p(z^{(1)}, t^{(1)}) - p(z^*, t^*)| + | p(z^*, t^*) - p(z^{(2)}, t^{(2)})|, \end{aligned} \tag{4.8}$$

which is simply an application of the result of Case 2 followed by an application of the result of Case 1. Combining the two cases results in the following estimates which hold for all cases.

LEMMA 4.4. *For Cases* I, II, *and* III, $0 \leqslant z^{(j)} \leqslant 1$ *and* $0 \leqslant t^{(j)} \leqslant t$, $j = 1, 2$,

$$\begin{aligned} &| p(z^{(1)}, t^{(1)}) - p(z^{(2)}, t^{(2)})| \\ &\quad \leqslant CC_2[7C + 8CKt + V] \cdot [| z^{(1)} - z^{(2)} | + | t^{(1)} - t^{(2)}|], \end{aligned} \tag{4.9a}$$

$$\begin{aligned} &| q(z^{(1)}, t^{(1)}) - q(z^{(2)}, t^{(2)})| \\ &\quad \leqslant CC_2[7C + 8CKt] \cdot [| z^{(1)} - z^{(2)}| + | t^{(1)} - t^{(2)}|]. \end{aligned} \tag{4.9b}$$

In order to obtain a priori estimates for the preservation of the Hölder classes of the derivatives of p and q under the mapping, we must consider the same three cases for the differentiated expressions in (2.5a), (2.5b), (2.6) and (2.7). The same type of application of the mean value theorem and the Lemmas 3.2 through 3.6 yields the following result which we state without proof.

LEMMA 4.5. *For* $0 \leqslant z^{(j)} \leqslant 1$ *and* $0 \leqslant t^{(j)} \leqslant t$, $j = 1, 2$,

$$\left| \frac{\partial p}{\partial z}(z^{(1)}, t^{(1)}) - \frac{\partial p}{\partial z}(z^{(2)}, t^{(2)}) \right| \leqslant C_3\{| z^{(1)} - z^{(2)} |^\alpha + | t^{(1)} - t^{(2)} |^\alpha\}, \tag{4.10a}$$

$$\left| \frac{\partial q}{\partial z}(z^{(1)}, t^{(1)}) - \frac{\partial q}{\partial z}(z^{(2)}, t^{(2)}) \right| \leqslant C_3\{| z^{(1)} - z^{(2)} |^\alpha + | t^{(1)} - t^{(2)} |^\alpha\}, \tag{4.10b}$$

$$\left| \frac{\partial p}{\partial t}(z^{(1)}, t^{(1)}) - \frac{\partial p}{\partial t}(z^{(2)}, t^{(2)}) \right| \leqslant C_4\{| z^{(1)} - z^{(2)} |^\alpha + | t^{(1)} - t^{(2)} |^\alpha\}, \tag{4.10c}$$

$$\left| \frac{\partial q}{\partial t}(z^{(1)}, t^{(1)}) - \frac{\partial q}{\partial t}(z^{(2)}, t^{(2)}) \right| \leqslant C_4\{| z^{(1)} - z^{(2)} |^\alpha + | t^{(1)} - t^{(2)} |^\alpha\}, \tag{4.10d}$$

where

$$C_3 = 2000\delta^{-2}C^6K^2V(2K + V)(\tilde{K}t + \tilde{V}), \tag{4.11}$$

and

$$C_4 = 5000\delta^{-3}C^6K^2(C + V)(\tilde{K}t + \tilde{V}). \tag{4.12}$$

Remark. Note that for the derivatives of p and q to be continuous, we must specify that the data satisfy the compatibility conditions.

$$\begin{aligned}\frac{\partial\zeta}{\partial t}(0, q_0(0), \varphi(0)) &+ \frac{\partial\zeta}{\partial q}(0, q_0(0), \varphi(0))\{R_2(0, 0, p_0(0), q_0(0))\\ &- \lambda_2(0, 0, p_0(0), q_0(0))\, q_0'(0)\} + \frac{\partial\zeta}{\partial w}(0, q_0(0), \varphi(0))\, v'(0)\\ &+ \{\lambda_1(0, 0, p_0(0), q_0(0))\, p_0'(0) - R_1(0, 0, p_0(0), q_0(0))\} = 0,\end{aligned} \tag{4.13}$$

where $v'(0)$ is evaluated through (1.1f) at $x = 0$, $t = 0$, and

$$\begin{aligned}\frac{\partial G}{\partial t}(0, p_0(1)) &+ \frac{\partial G}{\partial p}(0, p_0(1))\{R_1(1, 0, p_0(1), q_0(1)) - \lambda_1(1, 0, p_0(1), q_0(1))\, p_0'(1)\}\\ &- \{\lambda_2(1, 0, p_0(1), q_0(1))\, q_0'(1) - R_2(1, 0, p_0(1), q_0(1))\} = 0.\end{aligned} \tag{4.14}$$

5. A Priori Estimates for the Solution of the Parabolic Part of (1.1)

We begin our discussion with Case I of Section 1 where L and B are, respectively, the linear parabolic operator and linear boundary operator which are given by (1.2). For this case, we can assume that

$$S = S(x, t, p(0, t), q(0, t), w(x, t)) \tag{5.1}$$

which represents a mild generalization of the formulation in (1.1). In addition, we assume that the function $S = S(x, t, p, q, w)$ is defined on $\{(x, t, p, q, w): (x, t) \in \bar{Q}_T, -\infty < p < \infty, -\infty < q < \infty, -\infty < w < \infty\} = \bar{Q}_T \times \mathbb{R}^3$, that it possesses continuous first partial derivatives with respect to t, p, and q, that it possesses continuous fourth partial derivatives with respect to x and w, and that S and its aforementioned derivatives are bounded in absolute value by the positive constant M. With respect to the data $a_{ij} = a_{ij}(x, t)$, $i, j = 1, 2$, $b_i = b_i(x, t)$, $i = 1, 2$, we assume that they possess four continuous partial derivatives with respect to x and t in $\bar{Q}_T$ and that the functions and their derivatives are bounded in absolute value by M.

From (1.3) and the assumption that $\varphi(x)$ is bounded in absolute value by M, a direct application of the maximum principle for parabolic partial differential equations [4, 7, 9, 11] yields the following estimate.

LEMMA 5.1. *For* $(x, t) \in \bar{Q}_T$,

$$|w(x, t)| \leqslant (1 + t)M. \tag{5.2}$$

It is important to note at this point that the bound $\tilde{C}$ for v can be chosen to be $M(1 + T_1)$ for some fixed T_1.

Under the assumption that the initial data $\varphi(x)$ possesses fourth order continuous partial derivatives that are bounded in absolute value by M, and that the solution w is sufficiently smooth, the Method of Bernstein, which involves only x derivatives of (1.1f) [7, 9], can be applied directly to demonstrate the following result.

LEMMA 5.2. *For* $(x, t) \in \bar{Q}_T' = \{(x, t): x_1^2 + x_2^2 \leqslant r^2, 0 \leqslant t \leqslant T\}$, *where* r *is a positive number such that* $\bar{Q}_T' \subset Q_T$, *there exists a positive constant* M_1 *which depends only upon* M, T, *and* ν; *i.e.*, $M_1 = M_1(M, T, \nu)$, *such that*

$$\left| \frac{\partial w}{\partial x_i}(x, t) \right| \leqslant M_1, \qquad i = 1, 2, \tag{5.3a}$$

$$\left| \frac{\partial^2 w}{\partial x_i \, \partial x_j}(x, t) \right| \leqslant M_1, \qquad i, j = 1, 2, \tag{5.3b}$$

$$\left| \frac{\partial^3 w}{\partial x_i \, \partial x_j \, \partial x_k}(x, t) \right| \leqslant M_1, \qquad i, j, k = 1, 2, \tag{5.3c}$$

$$\left| \frac{\partial^4 w}{\partial x_i \, \partial x_j \, \partial x_k \, \partial x_\ell}(x, t) \right| \leqslant M_1, \qquad i, j, k, \ell = 1, 2. \tag{5.3d}$$

From the parabolic equation (1.1f), we can estimate $\partial w/\partial t$ in $\bar{Q}_T'$ via (5.3). Hence we see that the following is valid.

LEMMA 5.3. *For* $(x, t) \in \bar{Q}_T'$, *there exists a positive constant* M_2 *which depends only on* M_1, M, ν, *and* T, *i.e.*, $M_2 = M_2(M, T, \nu)$, *such that*

$$|(\partial w/\partial t)(x, t)| \leqslant M_2. \tag{5.4}$$

It is important to note at this point that the Lipschitz constant V *for* $v = v(t)$ *can be set equal to* M_2 *which is independent of the Lipschitz constant* K *of the functions* p *and* q *which will be determined in Section* 6.

By differentiating Eq. (1.1f) with respect to the space variables we can use (5.3) to estimate all of the mixed partial derivatives of w which involve one differentiation with respect to t. Finally, differentiating Eq. (1.1f) with respect to t, we see that we can obtain the following estimate for $\partial^2 w/\partial t^2$ in $\bar{Q}_T'$.

LEMMA 5.4. *For* $(x, t) \in \bar{Q}_T'$, *there exists a constant* M_3 *which depends only upon* M, T, ν *and* K, *i.e.*, $M_3 = M_3(M, T, \nu, K)$, *such that*

$$|(\partial^2 w/\partial t^2)(x, t)| \leqslant M_3. \tag{5.5}$$

It is important also to note at this point that, for this case the Hölder constant $\tilde{V}$ for $v = v(t)$ can be set equal to M_3 and $\alpha = 1$. Note also that M_3 is independent of $\tilde{K}$ which denotes the assumed Hölder constant for p and q and which will also be determined in Section 6.

Turning now to Case 2 of Section 1, we recall that (1.1f), (1.1g), (1.1h) become

$$\frac{\partial w}{\partial t} - \sum_{i,j=1}^{2} a_{ij}(x, t, w) \frac{\partial^2 w}{\partial x_i \, \partial x_j} + b(x, t, w, w_x) = S(x, t, p(0, t), q(0, t)), \quad (x, t) \in Q_T, \tag{5.6a}$$

$$\sum_{i,j=1}^{2} a_{ij}(x, t, w) \frac{\partial w}{\partial x_j} \cos(\mathbf{n}, x_i) = 0, \qquad (x, t) \in S_T, \tag{5.6b}$$

$$w(x, 0) = \varphi(x), \qquad x \in \Omega, \tag{5.6c}$$

where we assume that Ω is bounded and that it possesses a smooth boundary $\partial\Omega$. Let Ω_i, $i = 1, 2$, be bounded domains with smooth boundaries such that

$$\bar{\Omega} \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \tag{5.7}$$

and that the distance

$$d(\partial\Omega_1, \partial\Omega_2) \geqslant 1. \tag{5.8}$$

We split (5.6) into two problems. First let s denote the solution of

$$\frac{\partial s}{\partial t} - \sum_{i=1}^{2} \frac{\partial^2 s}{\partial x_i^2} = S(x, t, q(0, t), p(0, t)), \qquad (x, t) \in \Omega_2 \times (0, T], \tag{5.9a}$$

$$\frac{\partial s}{\partial n} = 0, \qquad (x, t) \in \partial\Omega_2 \times (0, T], \tag{5.9b}$$

$$s(x, 0) = 0, \qquad x \in \Omega_2, \tag{5.9c}$$

where we can assume with no loss of generality that S vanishes smoothly near $\partial\Omega_1$. Next we define the function r in $\bar{\Omega}$ via

$$r = w - s. \tag{5.10}$$

Substituting $r + s$ into (5.6) and using (5.9), we obtain

$$\frac{\partial r}{\partial t} - \sum_{i,j=1}^{2} a_{ij}(x, t, r + s) \frac{\partial^2 r}{\partial x_i \, \partial x_j} + b(x, t, r + s, r_x + s_x) + \sum_{i=1}^{2} \frac{\partial^2 s}{\partial x_i^2} - \sum_{i,j=1}^{2} a_{ij}(x, t, r + s) \frac{\partial^2 s}{\partial x_i \, \partial x_j} = 0, \quad (x, t) \in Q_T, \tag{5.11a}$$

$$\sum_{i,j=1}^{2} a_{ij}(x, t, r + s) \frac{\partial r}{\partial x_j} \cos(\mathbf{n}, x_i) + \sum_{i,j=1}^{2} a_{ij}(x, t, r + s) \frac{\partial s}{\partial x_j} \cos(\mathbf{n}, x_i) = 0, \quad (x, t) \in S_T, \tag{5.11b}$$

$$r(x, 0) = \varphi(x), \qquad x \in \Omega. \tag{5.11c}$$

Setting

$$\tilde{a}_{ij}(x, t, r) = a_{ij}(x, t, r + s), \tag{5.12a}$$

$$\tilde{b}(x, t, r, r_x) = b(x, t, r + s, r_x + s_x) + \sum_{i=1}^{2} \frac{\partial^2 s}{\partial x_i^2} - \sum_{i,j=1}^{2} a_{ij}(x, t, r + s) \frac{\partial^2 s}{\partial x_i \, \partial x_j}, \tag{5.12b}$$

$$\psi(x, t, r) = \sum_{i,j=1}^{2} a_{ij}(x, t, r + s) \frac{\partial s}{\partial x_j} \cos(\mathbf{n}, x_i), \tag{5.12c}$$

we obtain the second problem

$$\frac{\partial r}{\partial t} - \sum_{i,j=1}^{2} \tilde{a}_{ij}(x, t, r) \frac{\partial^2 r}{\partial x_i \, \partial x_j} + \tilde{b}(x, t, r, r_x) = 0, \qquad (x, t) \in Q_T, \tag{5.13a}$$

$$\sum_{i,j=1}^{2} \tilde{a}_{ij}(x, t, r) \frac{\partial r}{\partial x_j} \cos(\mathbf{n}, x_i) + \psi(x, t, r) = 0, \qquad (x, t) \in S_T, \tag{5.13b}$$

$$r(x, 0) = \varphi(x), \qquad x \in \Omega. \tag{5.13c}$$

We recall, for example, from [9, pp. 475–492] that if there exist positive constants ν, μ, μ_i, $i = 0, \ldots, 4$, such that

$$\nu \sum_{i=1}^{2} \xi_i^2 \leqslant \sum_{i,j=1}^{2} \tilde{a}_{ij}(x, t, r) \, \xi_i \xi_j \leqslant \mu \sum_{i=1}^{2} \xi_i^2, \tag{5.14a}$$

$$\left| \frac{\partial \tilde{a}_{ij}(x, t, r)}{\partial r}, \frac{\partial \tilde{a}_{ij}}{\partial x_k}, \psi, \frac{\partial \psi}{\partial r}, \frac{\partial \psi}{\partial x_k} \right| \leqslant \mu, \tag{5.14b}$$

$$| \tilde{b}(x, t, r, \mathbf{p})| \leqslant \mu \left(1 + \sum_{i=1}^{2} p_i^2\right), \tag{5.14c}$$

$$\left| \frac{\partial^2 \psi}{\partial r^2}(x, t, r), \frac{\partial^2 \psi}{\partial r \, \partial x_k}, \frac{\partial^2 \psi}{\partial r \, \partial t}, \frac{\partial \tilde{a}_{ij}}{\partial t}, \frac{\partial \psi}{\partial t} \right| \leqslant \mu, \tag{5.14d}$$

$$\left(\left| \frac{\partial \tilde{b}}{\partial p_1} \right|^2 + \left| \frac{\partial \tilde{b}}{\partial p_2} \right|^2\right)^{1/2} \left(1 + \sum_{i=1}^{2} p_i^2\right) + \left| \frac{\partial \tilde{b}}{\partial r} \right| + \left| \frac{\partial \tilde{b}}{\partial t} \right| \leqslant \mu \left(1 + \sum_{i=1}^{2} p_i^2\right), \tag{5.14e}$$

$$\left| \frac{\partial^2 \tilde{a}_{ij}}{\partial r^2}, \frac{\partial^2 \tilde{a}_{ij}}{\partial r \, \partial t}, \frac{\partial^2 \tilde{a}_{ij}}{\partial r \, \partial x_k}, \frac{\partial^2 \tilde{a}_{ij}}{\partial x_k \, \partial t} \right| \leqslant \mu, \tag{5.14f}$$

$$-r\tilde{b}(x, t, r, \mathbf{p}) \leqslant \mu_0 \sum_{i=1}^{2} p_i^2 + \mu_1 r^2 + \mu_2, \tag{5.14g}$$

$$-r\psi(x, t, r) \leqslant \mu_3 r^2 + \mu_4 \quad \text{for } (x, t) \in S_T, \tag{5.14h}$$

and if

the function $\partial\tilde{a}_{ij}/\partial x_k$ are Hölder continuous in the variables x with exponent β, $\partial\psi/\partial x_k$ are Hölder continuous in x and t with exponents β and $\beta/2$, respectively, and $\tilde{b}(x, t, r, \mathbf{p})$ is Hölder continuous in x with exponent β, (5.15a)

and

the initial datum $\varphi(x)$ satisfies the compatibility condition (5.15b)

$$\sum_{i,j=1}^{2} a_{ij}(x, 0, \varphi(x)) \frac{\partial \varphi}{\partial x_j}(x) \cos(\mathbf{n}, x_i) = 0 \qquad \text{for } x \in \partial\Omega,$$

then (5.13) possesses a classical solution r whose partial derivative with respect to t is Hölder continuous with exponent $\beta/2$ in $\bar{Q}_T$. Moreover the Hölder constant of $\partial r/\partial t$ depends only upon the constants ν, μ, μ_i, $i = 0,..., 4$, and the bounds upon the datum $\varphi(x)$ and its derivatives. Since the Bernstein estimates of Lemmas 5.1, 5.2, 5.3, and 5.4 hold for s in $\bar{\Omega}_1 \times [0, T]$, it follows from elementary calculations that if a_{ij}, $i = 1, 2$, and b satisfy (5.14) and (5.15), then there exist constants ν, μ, μ_i, $i = 0,..., 4$, which are independent of K such that the functions $\tilde{a}_{ij}$, $\tilde{b}$, and ψ satisfy (5.14) and (5.15). Combining the above result with the results of Lemmas 5.1 to 5.4 when applied to s, we obtain the following result.

LEMMA 5.5 (Quasi-linear Case). *For $(x, t) \in \bar{Q}_T'$, there exist positive constants M_4 and M_5 which are independent of K such that*

$$|w(x, t)| \leqslant M_4 \tag{5.16a}$$

$$|(\partial w/\partial t)(x, t)| \leqslant M_5, \tag{5.16b}$$

and there exists a positive constant M_6 which depends upon K, but not $\tilde{K}$, and a constant α, $0 < \alpha < 1$, such that for $(x, t^{(1)})$, $(x, t^{(2)}) \in \bar{Q}_T'$,

$$\left| \frac{\partial w}{\partial t}(x, t^{(1)}) - \frac{\partial w}{\partial t}(x, t^{(2)}) \right| \leqslant M_6 \, | t^{(1)} - t^{(2)} |^{\alpha}. \tag{5.17}$$

***Finally, it is important to note that for the quasi-linear case of* (1.1f), *the bound $\tilde{C}$ of v can be chosen to be M_4 and the Lipschitz constant V of $v = v(t)$ can be selected as M_5 while the Hölder constant $\tilde{V}$ of $v = v(t)$ can be selected as M_6. The choice of α for $v = v(t)$ is that of Lemma* 5.5.**

6. Continuity of the Mapping (2.9) and Its Preservation of Function Spaces

We begin our considerations with the hyperbolic part of the mapping $\mathscr{M}$. The first restriction on the time T occurred when we desired to limit our attention to the case of one bounce only off of a lateral boundary. This restriction amounted to

$$T < C^{-1}. \tag{6.1}$$

From a desire to simplify some of the constants, we restricted T via

$$T < (3CK)^{-1}. \tag{6.2}$$

Since Lemma 4.1 gurantees a uniform bound on the p and the q, we turn our consideration to the Lipschitz constants for p and q. From Lemmas 5.3 and 5.5 we see that V can be regarded as a constant independent of K. For preservation of the Lipschitz class for p and q we must select a K such that, for T suitably restricted,

$$7C^2C_2 + V + 8C^2C_2KT < K. \tag{6.3}$$

We must keep in mind that the value of C_2 displayed in (4.3) resulted from the restriction (6.2). We can handle both simultaneously by setting

$$K = 7C^2C_2 + V + 1 \tag{6.4}$$

and observing that this substituted into (6.3) leads to the restriction

$$T < (56C^4C_2^2 + 8C^2C_2V + 8C^2C_2)^{-1} \tag{6.5}$$

while the substitution of (6.4) into (6.2) leads to the restriction

$$T < (21C^3C_2 + 3CV + 3C)^{-1} \tag{6.6}$$

with is automatically satisfied when T is restricted as in (6.5). Consequently under this restriction for T, the Lipschitz class of p and q is preserved when K is chosen as that in (6.4). Turning now to the preservation of the Hölder classes of the first partial derivatives of p and q, we observe from Lemma 5.4 and 5.5 that the preservation of the Hölder constant $\tilde{K}$ amounts to the selection of $\tilde{K}$ so that a suitable restriction of T yields

$$C_5\tilde{K}T + C_6 < \tilde{K}, \tag{6.7}$$

where C_5 and C_6 can be seen to depend only upon C, C_2, V, $\tilde{V}$, and δ. Selecting

$$\tilde{K} = C_6 + 1 \tag{6.8}$$

and restricting T to satisfy

$$T < \{C_5(C_6 + 1)\}^{-1}, \tag{6.9}$$

we ensure that the Hölder classes of the first partial derivatives of p and q are preserved under the mapping $\mathscr{M}$, provided that α is selected from either Lemma 5.4 or Lemma 5.5 depending upon which parabolic operator is used.

As the choice of $\tilde{C}$ and V could be made independently of K, the Lipschitz class of $v = v(t)$ is preserved under the mapping $\mathscr{M}$. Following the determination of K by (6.4) it follows from Lemmas 5.4 and 5.5 that for the choice of $\tilde{V}$ and α indicated by those results, we see that the Hölder class of dv/dt is preserved under the mapping. Thus the mapping $\mathscr{M}$ takes a compact and convex subset of the Banach space of continuous functions on $[0, T]$ with the uniform norm topology into itself. All that remains to achieve existence is to demonstrate that the mapping $\mathscr{M}$ is continuous.

We define

$$\| f \|_t = \sup_{\substack{0 \leqslant z \leqslant 1 \\ 0 \leqslant \tau \leqslant t}} | f(z, \tau)| \tag{6.10}$$

and

$$||| v |||_t = \sup_{0 \leqslant \tau \leqslant t} | v(\tau)| . \tag{6.11}$$

Let (p_j , q_j), $j = 1, 2$, denote solutions of the hyperbolic part of (1.1) which correspond, respectively, to the data v_j , $j = 1, 2$. Using Lemma 3.1 and the elementary techniques of Section 3, we obtain the estimate

LEMMA 6.1. *For* $0 \leqslant z \leqslant 1$ *and* $\max(0, t_i) \leqslant \tau \leqslant t \leqslant T$,

$$| z_i^{(1)}(\tau; z, t) - z_i^{(2)}(\tau; z, t)| \leqslant C \exp\{3CKt\}\, t\{\| p_1 - p_2 \|_t + \| q_1 - q_2 \|_t\}, \quad i = 1, 2, \tag{6.12}$$

where $z_i^{(j)}$ *corresponds to the ith characteristic determined by* (p_j , q_j), $j = 1, 2$.

Differencing the formulas involving $t_i^{(j)}(z, t)$ that are obtained by substituting the $(p_j , q_j), j = 1, 2$, into (3.6), we can use the technique of proof of Lemma 3.3 to obtain the following result.

LEMMA 6.2. *For* $0 \leqslant z \leqslant 1$ *and* $0 \leqslant \min(t_i^{(1)}, t_i^{(2)}) \leqslant t \leqslant T$,

$$| t_i^{(1)}(z, t) - t_i^{(2)}(z, t)| \leqslant \delta^{-1}[(C + 2CK)\, tC \exp\{3CKt\} + C]\, t\{\| p_1 - p_2 \|_t + \| q_1 - q_2 \|_t\}. \tag{6.13}$$

In order now to study the continuous dependence of the p_j and q_j on v_j , $j = 1, 2$, we must use Lemmas 6.1 and 6.2 to assist in estimating the differences of the formulas (2.5a), (2.5b), (2.6), and (2.7) into which we substitute p_j , q_j , and v_j . The estimation is elementary and yields the following estimate.

LEMMA 6.3. *There exists a constant* C_7 *which depends only upon* C, K, *and* V *such that for* $0 \leqslant t \leqslant T$,

$$\| p_1 - p_2 \|_t + \| q_1 - q_2 \|_t \leqslant C_7 t\{\| p_1 - p_2 \|_t + \| q_1 - q_2 \|_t\} + C ||| v_1 - v_2 |||_t , \tag{6.14}$$

and if T is restricted by

$$T < (2C_7)^{-1}, \tag{6.15}$$

then

$$\| p_1 - p_2 \|_t + \| q_1 - q_2 \|_t \leqslant 2C \| v_1 - v_2 \|_t . \tag{6.16}$$

We complete our consideration of the continuity of the mapping $\mathscr{M}$ by estimating the continuous dependence of w upon p and q via the maximum principle and applying (6.16).

Consider the case of the linear parabolic operator and linear boundary operator of (1.2) coupled with the S given by (5.1). Letting w_j correspond to the data (p_j, q_j), $j = 1, 2$, and setting $\eta = w_1 - w_2$, we difference the corresponding equations (1.1f), (1.1g), and (1.1h) and obtain

$$\eta_t - \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 \eta}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i \frac{\partial \eta}{\partial x_i} - \frac{\partial S}{\partial w} \eta = \frac{\partial S}{\partial p} (p_1 - p_2) + \frac{\partial S}{\partial q} (q_1 - q_2), \qquad (x, t) \in Q_T, \tag{6.17a}$$

$$\sum_{i,j=1}^{2} a_{ij} \frac{\partial \eta}{\partial x_j} \cos(\mathbf{n}, x_i) = 0, \qquad (x, t) \in S_T, \tag{6.17b}$$

$$\eta(x, 0) = 0, \qquad x \in \Omega, \tag{6.17c}$$

where the functions $\partial S/\partial w$, $\partial S/\partial p$, and $\partial S/\partial q$ are evaluated at the usual intermediate points indicated by the mean value theorem. Constructing a function $\psi(x)$ which is positive and twice continuously differentiable in Ω and which satisfies

$$\sum_{i,j=1}^{n} a_{ij}(\partial\psi/\partial x_j) \cos(\mathbf{n}, x_i) > 0 \qquad \text{for } x \in \partial\Omega, \tag{6.18}$$

it follows from $| \partial S/\partial w | \leqslant M$ that for

$$\xi(x, t) = \eta(x, t) \exp\{Mt\} \tag{6.19}$$

we can apply the usual maximum principle argument [4, 7, 9, 11] to show that for each $\epsilon > 0$,

$$| \xi(x, t)| \leqslant \epsilon\psi(x) + C_8 t \tag{6.20}$$

provided that

$$C_8 = \exp\{MT\} M\{\| p_1 - p_2 \|_t + \| q_1 - q_2 \|\} + \epsilon C_9 , \tag{6.21}$$

where C_9 is a positive constant which depends on the bounds of ψ and its first and second partial derivatives. Taking $\epsilon = 0$ in (6.20) and (6.21) and using (6.19) to replace ξ in (6.20), we obtain from (6.16) the following result.

LEMMA 6.4. *For* $0 \leqslant t \leqslant T$,

$$||| w_1(0, \cdot) - w_2(0, \cdot)|||_t \leqslant 2CM \exp\{MT\}t ||| v_1 - v_2 |||_t . \qquad (6.22)$$

For the quasi-linear case of the parabolic part of (1.1), a similar argument yields the following similar result.

LEMMA 6.5 (Quasi-linear Case). *For* $0 \leqslant t \leqslant T$, *there exists a positive constant* C_{10} *which depends only upon* M *and the data for the operator* L *such that*

$$||| w_1(0, \cdot) - w_2(0, \cdot)|||_t \leqslant C_{10}t ||| v_1 - v_2 |||_t . \qquad (6.23)$$

Now, the restriction (6.15) is precisely the one which guarantees that the hyperbolic part of (1.1) is uniquely solvable for a given v. Consequently, under all of the aforementioned restrictions on T, the mapping $\mathscr{M}$ is well defined and the continuity of $\mathscr{M}$ follows from Lemma 6.4 or Lemma 6.5. The Schauder fixed point theorem guarantees the existence of a fixed point v and thus a solution of (1.1). This solution is either weak or classical depending upon whether one wishes to carry all of the data and time restrictions necessary to ensure the preservation of the Hölder classes under the mapping $\mathscr{M}$. To obtain uniqueness we need to make one additional restriction upon the time T so that either

$$2CM \exp\{MT\}T < 1 \qquad (6.24)$$

or

$$C_{10}T < 1. \qquad (6.25)$$

With the appropriate one of these restrictions for the parabolic case considered, the mapping $\mathscr{M}$ becomes a contraction and the fixed point becomes unique.

7. SUMMARY OF RESULTS

The following statement of results contains the sense of the foregoing analysis and it avoids a catalog of assumptions upon the data.

THEOREM. *If the parabolic part of the nonlinear hyperbolic–parabolic system of* (1.1) *possesses a classical solution when Lipschitz continuous* $p(0, t)$ *and* $q(0, t)$ *are substituted into* S, *then for* T *sufficiently small, but positive, the nonlinear hyperbolic–parabolic system* (1.1) *possesses a unique weak solution or a unique classical solution depending upon whether the hyperbolic part of* (1.1) *possesses a unique generalized solution or a unique classical solution when* $w(0, t)$ *in* ζ *is replaced by a function* $v = v(t)$ *which possesses a Hölder continuous first derivative with respect to* t.

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