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Determination of a Source Term in a Linear Parabolic Partial Differential Equation

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1. Introduction

The purpose of this paper is to generalize a result of one of the authors [7] for the heat equation to more general parabolic equations. Suppose that an unknown heat source f, which is dependent only on the spatial variable, is operating on the unit interval. Suppose the usual initial and boundary temperature data are overspecified by the additional measurement of the heat flow rate at the boundary $x = 0$ over an interval of time $0 \leq \tau \leq t \leq T$, $T > 0$.

The mathematical problem can be stated as: find the pair $u = u(x, t)$ and $f = f(x)$ that satisfies

$$Lu \equiv (p(x)u_x)_x - q(x)u_t = \rho(x)f(x), \qquad 0 < x < 1, \qquad 0 < t \leq T, \tag{1.1a}$$

$$u(0, t) = \sigma_1(t), \qquad 0 < t \leq T, \tag{1.1b}$$

$$u(1, t) = \sigma_2(t), \qquad 0 < t \leq T, \tag{1.1c}$$

$$u(x, 0) = \sigma_3(x), \qquad 0 \leq x \leq 1, \tag{1.1d}$$

$$p(0)u_x(0, t) = g(t), \qquad 0 \leq \tau \leq t \leq T, \tag{1.1e}$$

where $\sigma_1(t)$, $\sigma_2(t)$, $\sigma_3(x)$ and $g(t)$ are known functions. We achieve the form in (1.1a) from the form of a general equation

$$C(x)u_{xx} + E(x)u_x + F(x)u - u_t = f(x) \tag{1.2}$$

by the use of the integrating factor $\rho(x)$ as in Petrovskii [10].

Using the linearity of L and the same arguments as in [7] we obtain uniqueness. The example presented in the introduction of [7] demonstrates that this problem is not well posed in the sense of Hadamard. Hence, a priori information concerning f is necessary in order to insure continuous dependence of the solution (u, f) upon the data σ_i, $i = 1, 2, 3$, and g.

The solution (u, f) can be written in the form $(u, f) = (v, 0) + (z, f)$, where v satisfies

$$\begin{aligned} &Lv = 0, \quad 0 < x < 1, \quad 0 < t \le T, \\ &v(0, t) = \sigma_1(t), \quad 0 < t \le T, \\ &v(1, t) = \sigma_2(t), \quad 0 < t \le T, \\ &v(x, 0) = \sigma_3(x), \quad 0 \le x \le 1, \end{aligned} \tag{1.3}$$

where (z, f) satisfies

$$Lz = \rho f, \quad 0 < x < 1, \quad 0 < t \le T, \tag{1.4a}$$

$$z(0, t) = z(1, t) = 0, \quad 0 < t \le T, \tag{1.4b}$$

$$z(x, 0) = 0, \quad 0 \le x \le 1, \tag{1.4c}$$

$$p(0) \frac{\partial z}{\partial x}(0, t) = G(t), \quad 0 \le \tau \le t \le T, \tag{1.4d}$$

and where

$$G(t) = g(t) - p(0) \frac{\partial v}{\partial x}(0, t). \tag{1.5}$$

Since v and $\partial v / \partial x$ depend continuously upon the σ_i, $i = 1, 2, 3$, it will suffice to show that (z, f) depends continuously upon G. Hence, we shall restrict our attention to (1.4) and its solution (z, f).

It is convenient here to list some hypotheses that will be needed below.

Assumption 1. We shall assume that f is twice continuously differentiable in $0 \le x \le 1$, that $f(0) = f(1) = 0$, and that there exists a positive constant K, such that f and its first and second derivatives with respect to x, f' and f'', are uniformly bounded in absolute value by K.

Assumption 2. The functions p, p', q and ρ are uniformly Hölder continuous in $0 \le x \le 1$ and satisfy

$$0 < p_* \le p(x) \le p^*, \tag{1.6a}$$

$$0 < \rho_* \le \rho(x) \le \rho^*, \tag{1.6b}$$

$$0 \le q_* \le q(x) \le q^*, \tag{1.6c}$$

and

$$|p'(x)| \le p'^*. \tag{1.6d}$$

Remark. We shall refer to the constants in (1.6) either individually or collectively through the symbol $\mathscr{D}$ which denotes the set of constants in (1.6). Also, in what follows we shall denote

$$\|f\|_{[a,b]} = \sup_{a \le \xi \le b} |f(\xi)|$$

for any real valued function f defined on $a \leq \xi \leq b$, and we shall denote

$$\|f\|_2 = \left(\int_0^1 [f(x)]^2 dx\right)^{1/2}$$

for any real valued function f defined on $0 \leq x \leq 1$. It is convenient here to mention that positive constants will be denoted by a subscripted K and the dependence upon various parameters will be displayed via the usual function notation; e.g. $K_{10^6} = K_{10^6}(\mathscr{D})$.

Theorem. *If assumptions 1 and 2 hold, then for each $N > 0$, there exist families of constants $A_N^{(i)}$ and $B_N^{(i)}$ $i = 1, 2$ such that $\lim_{N\to\infty} A_N^{(i)} = \infty$ and $\lim_{N\to\infty} B_N^{(i)} = 0$ and there exists a constant ν, $0 < \nu < 1$, such that*

$$\|f\|_2^2 \leq A_N^{(1)}\eta^\nu + B_N^{(1)}, \tag{1.7}$$

and

$$\|z(\cdot, t)\|_2^2 \leq A_N^{(2)}\eta^\nu + B_N^{(2)}, \tag{1.8}$$

where

$$\eta = \|G\|_{[t,T]} \tag{1.9}$$

and (z, f) is the solution of (1.4).

2. Proof of the Theorem

The assumed smoothness of f allows us to expand it into the series form

$$f(x) = \sum_{n=1}^{\infty} c_n\psi_n(x), \tag{2.1}$$

where the ψ_n are the normalized eigenfunctions corresponding to the eigenvalues λ_n for the Sturm-Liouville problem

$$\begin{aligned} &(p\psi_n')' - q\psi_n + \rho\lambda_n\psi_n = 0, \qquad 0 < x < 1,\\ &\psi_n(0) = \psi(1) = 0, \end{aligned} \tag{2.2}$$

and where

$$c_n = \int_0^1 \rho(x)f(x)\psi_n(x)dx \tag{2.3}$$

are the Fourier coefficients of f. We should note here that the λ_n are positive and form a monotone increasing sequence such that $\lim_{n\to\infty} \lambda_n = \infty$. An estimate for f can be obtained from an estimate of the c_n. Standard separation of variables yields the formal representation

$$z(x, t) = \sum_{n=1}^{\infty} (-1)c_n[\lambda_n^{-1} - \lambda_n^{-1} \exp\{-\lambda_n t\}]\psi_n(x) \tag{2.4}$$

for z in terms of the Fourier coefficients of f.

From Assumption 1 and the Hilbert-Schmidt theorem [9], it follows that there exists a positive constant $K_1 = K_1(K, \mathscr{D})$ so that

$$|c_n| \leq K_1\lambda_n^{-1}. \tag{2.5}$$

From [3] and [10] we note that

$$(\rho^*)^{-1}(p_*\pi^2 n^2 + q_*) \leq \lambda_n \leq (\rho_*)^{-1}(p^*\pi^2 n^2 + q^*), \tag{2.6}$$

$$|\psi_n(x)| \leq p_*^{-1/2}\lambda_n^{1/2}, \qquad 0 \leq x \leq 1, \tag{2.7}$$

and

$$|\psi_n'(x)| \leq p_*^{-1}(p^*\lambda_n)^{1/2} + 2p_*^{-1}\rho_*^{-1/2}(\lambda_n\rho^* + q^*). \tag{2.8}$$

Consequently, the formal representation for z converges absolutely and uniformly and its partial derivative with respect to x can be obtained by differentiating term by term. We see that this yields

$$F(t) \equiv p(0)\frac{\partial z}{\partial x}(0, t) = \sum_{n=1}^{\infty} c_n\lambda_n^{-1}p(0)\psi_n'(0)[\exp\{-\lambda_n t\} - 1]. \tag{2.9}$$

From (1.4d), we see that

$$|F(t)| \leq \eta, \qquad 0 \leq \tau \leq t \leq T, \tag{2.10}$$

where η is defined by (1.9).

Let $\zeta = t + i\beta$. Clearly, $F(\zeta)$ is an analytic function in the complex domain $\operatorname{Re}\zeta \geq 0$. Moreover, there exists a positive constant $K_2 = K_2(\mathscr{D}, K)$ such that for all ζ in $\operatorname{Re}\zeta \geq 0$,

$$|F(\zeta)| \leq K_2. \tag{2.11}$$

Employing arguments similar to those in [4, 5, 6] which involve applications of a lemma of Lindelof [4] and Carleman [4], it follows that there exists a constant $\alpha = \alpha(\tau, T, \tau^*)$, $0 < \alpha < 1$ and a positive constant $K_3 = K_3(\mathscr{D}, K)$ such that for all ζ satisfying $\frac{3}{4}\tau + \frac{1}{4}T \leq \operatorname{Re}\zeta \leq \frac{1}{4}\tau + \frac{3}{4}T$ and $|\operatorname{Im}\zeta| \leq 2\tau^*$,

$$|F(\zeta)| \leq K_3\eta^\alpha. \tag{2.12}$$

An elementary estimation of the Cauchy-Riemann representation formula for $F'(\zeta)$ yields

$$\left|F'\left(\frac{\tau + T}{2} + i\beta\right)\right| \leq K_4\eta^\alpha, \qquad |\beta| \leq \tau^*, \tag{2.13}$$

where $K_4 = K_4(\mathscr{D}, K, \tau, \tau^*, T)$ from (2.9) we see that

$$F'\left(\frac{\tau + T}{2} + i\beta\right) = \sum_{n=1}^{\infty} a_n \exp\{-i\lambda_n\beta\}, \tag{2.14}$$

where

$$a_n = (-1)c_n p(0)\psi_n'(0)\exp\{-\lambda_n 2^{-1}(\tau + T)\}. \tag{2.15}$$

Utilizing a lemma of Binmore [1, 2] we obtain

$$|a_n| \leq H(\lambda_n)K_4\eta^\alpha, \tag{2.16}$$

where

$$H(\lambda_n) = \left\{\prod_{k=1}^{n-1} \cos\left(\frac{\pi}{2}\frac{\lambda_k}{\lambda_n}\right) \prod_{k=n+1}^{\infty} \cos\left(\frac{\pi}{2}\frac{\lambda_n}{\lambda_k}\right)\right\}^{-1} \tag{2.17}$$

and $\tau^* = \tau^*(\mathscr{D})$ is selected so that it exceeds every

$$\tau_n = \frac{\pi}{2}\left\{\frac{n}{\lambda_n} + \sum_{k=n+1}^{\infty} \lambda_k^{-1}\right\}, n = 1, 2, \ldots. \tag{2.18}$$

We shall be able to estimate f via the c_n after we have obtained a lower estimate of the absolute value of $\psi_n'(0)$. To this end we consider the solution (X, φ) of

$$\begin{aligned} &X' = \varphi, \qquad 0 \leq x \leq 1,\\ &\varphi' = -p'(x)[p(x)]^{-1}\varphi - (\lambda_n\rho - q)[p]^{-1}X, \qquad 0 \leq x \leq 1,\\ &X(0) = 0,\\ &\varphi(0) = 1. \end{aligned} \tag{2.19}$$

Clearly,

$$\psi_n'(0) = \left(\int_0^1 \rho(x)[X(x)]^2 dx\right)^{-1/2}. \tag{2.20}$$

Consequently, a lower estimate can be obtained from an upper estimate of X. Set

$$w(x) = \max(\|X\|_{[0,x]}, \|\varphi\|_{[0,x]}). \tag{2.21}$$

Then, it follows that there exists a constant $K_5 = K_5(n, \mathscr{D})$ such that

$$w(x) \leq 1 + K_5 \int_0^x w(\xi)d\xi. \tag{2.22}$$

An application of Gronwall's lemma yields $w(x) \leq \exp\{K_5\}$ and

$$\|X\|_{[0,1]} \leq \exp\{K_5\}. \tag{2.23}$$

Hence, (2.20) and (2.23) yield

$$|\psi_n'(0)| \geq \rho^{*1/2} \exp\{-K_5\}. \tag{2.24}$$

Using (1.6), (2.15), (2.16), (2.17) and (2.24), we obtain

$$|c_n| \leq p_*^{-1}\rho^{*1/2} \exp\{2^{-1}(\tau + T)\lambda_n + K_5\}\cdot H(\lambda_n)K_4\eta^\alpha \tag{2.25}$$

which we proceed to use in estimating the L^2 norm of f.

From (2.1) and the orthogonality of the ψ_n, we have

$$\int_0^1 \rho(x)[f(x)]^2 dx = \sum_{n=1}^{\infty} c_n^2 \tag{2.26}$$

which implies that

$$\|f\|_2^2 \leq \rho_*^{-1} \sum_{n=1}^{\infty} c_n^2. \tag{2.27}$$

For any $N > 0$,

$$\|f\|_2^2 \leq \rho_*^{-1} \sum_{n=1}^{[N]} c_n^2 + \rho_*^{-1} \sum_{n=[N]+1}^{\infty} c_n^2. \tag{2.28}$$

Recalling (2.5) and (2.25), it follows from (2.28) that

$$\|f\|_2^2 \leq A_N \eta^{2\alpha} + B_N, \tag{2.29}$$

where A_N and B_N are a family of positive constants such that $\lim_{N\to\infty} A_N = \infty$ and $\lim_{N\to\infty} B_N = 0$. We see from (2.4), (2.26) and (1.6) that there exists a positive constant $K_6 = K_6(\mathscr{D})$ such that

$$\|z(\cdot, t)\|_2^2 \leq K_6 \|f\|_2^2.$$

Hence, the theorem is valid.

3. An Asymptotic Estimate as $\eta \to 0$

A not-well-posed problem in the sense of Hadamard is computationally feasible only if the dependence of the solution upon the data is either Hölder ($0(\eta^\beta)$, $0 < \beta \leq 1$) or logarithmic

$$\left(0\left(\frac{1}{\left(\log \frac{1}{\eta}\right)^\beta}\right)\right).$$

In fact, logarithmic dependence requires such data accuracy as to border on the impractical. Consequently it is of interest to study the forms $A_N^{(i)}\eta^\nu + B_N^{(i)}$ and to estimate their minimums over $N > 0$. This will yield a function of η which will give some estimate of the type of continuity to expect from (1.4).

We begin our study with $A_N^{(1)}$. Note that from (2.28) and (2.25) we see that to estimate $A_N^{(i)}$ we must estimate $H(\lambda_n)$. Recalling the form of $H(\lambda_n)$ from (2.17), we consider first the question of when

$$0 \leq \frac{\pi}{2}\frac{\lambda_n}{\lambda_k} \leq \frac{\pi}{4}.$$

This is equivalent to $\lambda_k > 2\lambda_n$. From (2.6) it follows from some elementary calculations that there exist positive constants $K_7 = K_7(\mathscr{D})$ and $K_8 = K_8(\mathscr{D})$ so that for

$$k \geq N_n = (K_7 n^2 + K_8)^{1/2} \tag{3.1}$$

we have

$$0 \leq \frac{\pi}{2}\frac{\lambda_n}{\lambda_k} \leq \frac{\pi}{4}. \tag{3.2}$$

Next, we estimate

$$\prod_{k=N_n}^{\infty} \cos\left(\frac{\pi}{2}\frac{\lambda_n}{\lambda_k}\right) = \exp\left\{\sum_{k=N_n}^{\infty} \log\cos\frac{\pi}{2}\frac{\lambda_n}{\lambda_k}\right\}$$

by noting that

$$\log\cos\frac{\pi}{2}\frac{\lambda_n}{\lambda_k} = -\int_0^{\pi/2(\lambda_n/\lambda_k)} \tan\xi\, d\xi \geq -\frac{\pi}{2}\frac{\lambda_n^2}{\lambda_k^2}$$

and utilizing (2.6) along with some simple calculations to obtain

$$\prod_{k=N_n}^{\infty} \cos\frac{\pi}{2}\frac{\lambda_n}{\lambda_k} \geq \exp\{-K_9 n^2 - K_{10}\}, \tag{3.3}$$

where $K_9 = K_9(\mathscr{D})$ and $K_{10} = K_{10}(\mathscr{D})$. Let

$$d = \inf_n (\lambda_{n+1} - \lambda_n) \tag{3.4}$$

and *assume that* $d > 0$. We can finish the estimate of $H(\lambda_n)$ by noting that

$$\prod_{k=1}^{n-1} \cos\left(\frac{\pi}{2}\frac{\lambda_k}{\lambda_n}\right) \prod_{k=n+1}^{N_k-1} \cos\frac{\pi}{2}\frac{\lambda_n}{\lambda_k} \geq \left\{\min\left(\cos\frac{\pi}{2}\frac{\lambda_n - d}{\lambda_n}, \cos\frac{\pi}{2}\frac{\lambda_n}{\lambda_n + d}\right)\right\}^{N_n}. \tag{3.5}$$

Using (2.6) again, we obtain a lower estimate of the right hand side of (3.5) which we combine with (3.3) and (2.17) to obtain

$$H(\lambda_n) \leq d^{-N_n}(K_{11}n^2 + K_{12})^{N_k} \exp\{K_9 n^2 + K_{10}\}, \tag{3.6}$$

where $K_{11} = K_{11}(\mathscr{D})$ and $K_{12} = K_{12}(\mathscr{D})$. It is not difficult to obtain

$$H(\lambda_n) \leq \exp\{K_{13}n^2 + K_{14}\}, \tag{3.7}$$

where $K_{13} = K_3(\mathscr{D}, d)$ and $K_{14} = K_{14}(\mathscr{D}, d)$. Looking at (2.25) it is not difficult to see that

$$A_N^{(1)} \leq \exp\{K_{15}N^2 + K_{16}\}, \tag{3.8}$$

where $K_{15} = K_{15}(\mathscr{D}, \tau, T, K, d)$ and $K_{16} = K_{16}(\mathscr{D}, \tau, T, K, d)$. Clearly, a similar expression holds for $A_N^{(2)}$.

With respect to $B_N^{(1)}$, it is clear that from (2.5), (2.6) and (2.28) that we have

$$B_N^{(1)} \leq K_{17}N^{-3}, \tag{3.9}$$

where $K_{17} = K_{17}(\mathscr{D}, K)$. Likewise, we see that a similar estimate holds for $B_N^{(2)}$.

Now, the inequalities (3.8) and (3.9) yield

$$A_N^{(1)}\eta^\nu + B_N^{(1)} \leq \exp\{K_{15}N^2 + K_{16}\}\eta^\nu + K_{17}N^{-3}. \tag{3.10}$$

Setting

$$\exp\{K_{15}N^2 + K_{16}\}\eta^\nu = K_{17}N^{-3},$$

we see that

$$K_{15}N^2 + 3\log N + K_{16} - \log K_{17} = \nu \log \frac{1}{\eta}.$$

As $\eta \to 0$, $N = 0\left(\log \frac{1}{\eta}\right)$ and

$$\min_{N>0} (A^{(1)}\eta^\nu + B_N^{(1)}) = 0\left(\frac{1}{\left(\log \frac{1}{\eta}\right)^3}\right). \tag{3.11}$$

Since similar estimates hold for $A_N^{(2)}\eta^\nu + B_N^{(2)}$, we have established the following result.

Theorem. *If assumptions 1 and 2 hold and if there exists a minimum positive separation between the eigenvalues of the Sturm-Liouville problem* (2.2), *then the* L^2 *norms of the components of the solution* (z, f) *of* (1.4) *depend logarithmically upon the uniform of the data G.*

4. Comments on Numerical Procedures

Logarithmic dependence upon the data implies that any numerical method for (1.4) would be marginal at best. Any formulation for (1.4) would involve a mathematical programming problem of the least squares or linear programming variety. For examples of these types of formulations we refer the reader to [6, 7].

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Abstract

The determination of a spatially dependent source term in a linear parabolic differential equation whose coefficients depend only on the space variable from the specification of the heat flux at one of the boundaries in addition to the usual boundary-initial conditions is a not well-posed problem in the sense of Hadamard. Continuous dependence upon the data is studied and it is demonstrated that the asymptotic dependence as the norm of the data tends to zero is no worse than logarithmic.

Zusammenfassung

Wir betrachten eine lineare parabolische Differentialgleichung, deren Koeffizienten zeitunabhängig sind. Die Bestimmung der unbekannten zeitunabhängigen Wärmequelle aus dem seitlichen Wärmefluss, zusätzlich zu den üblichen Randbedingungen ist kein wohlbestimmtes Problem im Sinne von Hadamard. Wir betrachten die stetige Abhängigkeit der Lösungen von diesen Daten und zeigen, dass die asymptotische Abhängigkeit höchstens logarithmisch ist, wenn die Norm der Daten gegen Null strebt.

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